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ON THE CHARACTERISTIC CLASSES OF MASLOV–ARNOLD

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In the present note we discuss the characteristic classes of Lagrangian manifolds introduced by V. P. Maslov ([1]) and V. I. Arnold ([2])*. We shall give an expression for these classes in terms of the Stiefel–Whitney and A. Borel classes.

§ 1. **Notation. Definitions.** By $E = E^{2n}$ we denote the space of real coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$. Define the transformation $J : E \to E$ by the formula $J(p_1, \dots, p_n, q_1, \dots, q_n) = (-q_1, \dots, -q_n, p_1, \dots, p_n)$. Clearly, $J^2 = -1$. We shall call an n-dimensional hyperplane $\Pi \subset E$ **Lagrangian** if the plane $J(\Pi)$ is orthogonal to it. The totality of (nonoriented) Lagrangian planes of the space E passing through the origin of coordinates will be denoted by Λ_n. The manifold Λ_n is diffeomorphic to the quotient space $U(n)/O(n)$ of the group of unitary matrices by the group of orthogonal matrices.

Let X be a finite CW-complex. We shall say that an n-dimensional Lagrangian bundle with base X is given if the following are given: (1) an $O(n)$-bundle ξ with base X; (2) an equivalence of the complexification $c\xi$ of the bundle ξ and of the trivial $U(n)$-bundle with base X. We always denote a Lagrangian bundle by the same letter as the $O(n)$-bundle defining it. It is easy to see that *classes of equivalent n-dimensional Lagrangian bundles with base X are in one-to-one correspondence with homotopy classes of maps X into Λ_n* (where the identity map $\Lambda_n \to \Lambda_n$ corresponds to the bundle induced by the principal fibration $U(n) \to U(n)/O(n) = \Lambda_n$).

Let us note the following important example. Let $M^n \subset E$ be a smooth submanifold of dimension n. The manifold M^n is called **Lagrangian** if all its tangent planes are Lagrangian. The complexification of any Lagrangian plane $\Pi \subset E$ is canonically isomorphic to E; therefore the tangent bundle to a Lagrangian manifold may be regarded as a Lagrangian bundle. The corresponding map $M^n \to \Lambda_n$ assigns to a point $x \in M^n$ the plane $\Pi \subset E$ passing through the origin of the coordinate space E and parallel to the tangent plane to the manifold M^n at the point x.

§ 2. **Subsets of the manifold M^n.** Let $l \leq k \leq n$; P_k is the linear span of the vectors $p_1, p_2, \dots, p_k$. Denote by $\Lambda_n^{k,l}$ the subset of the manifold Λ_n consisting of all Lagrangian planes whose intersection with P_k is at least l-dimensional. The sets $\Lambda_n^{k,l}$ are not submanifolds of Λ_n, but they can be represented in the following way as images of manifolds. Denote by $\widetilde{\Lambda}_n^{k,l}$ the space of pairs (Π, Π'), where $\Pi \in \Lambda_n$, and Π' is an l-dimensional plane of the space E such that $\Pi' \subset \Pi \cap P_k$. The manifold $\widetilde{\Lambda}^{k,l}$ is the space of a smooth fibration with base $G_{k,l}$ (the Grassmann manifold) and fiber Λ_{n-l}. The projection into

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* In these papers only the first (one-dimensional) of the classes considered here was introduced. The definition of the remaining classes was given by V. I. Arnold in his seminar.

in this fibration takes (Π, Π') to Π'. The mapping $i = i_n^{k,l} : \widetilde{\Lambda}_n^{k,l} \to \Lambda_n$, where $i(\Pi, \Pi') = \Pi$, has as its image $\Lambda_n^{k,l}$. On $i^{-1}(\Lambda_n^{k,l} \setminus \Lambda_n^{k,l+1})$ it is a diffeomorphism onto $\Lambda_n^{k,l} \setminus \Lambda_n^{k,l+1}$; the singularities of this mapping are concentrated on the set $i^{-1}(\Lambda_n^{k,l+1})$, which, as is easily seen, has codimension $n-k+1$. In what follows we shall mainly have to consider the sets $A_k = \Lambda_n^{n-k+1,1}$, $B_k = \Lambda_n^{n-k+1,2}$, and $\Lambda_n^k = \Lambda_n^{n,k}$. Let us note that the dimension of the manifold Λ_n is $n(n+1)/2$, and the codimensions of A_k, B_k, and Λ_n^k in Λ_n are respectively k, $2k+1$, and $k(k+1)/2$.

§ 3. **Orientability. Cohomology classes.** The manifold Λ_n is orientable if and only if n is odd. In order that the manifold $\widetilde{\Lambda}_n^{k,l}$ be orientable, it is necessary and sufficient that the fiber Λ_{n-l} be orientable and that the base $G_{k,l}$ and the fibration itself be orientable or nonorientable simultaneously. The fiber Λ_{n-l} is orientable if and only if $n-l$ is odd, and the base $G_{k,l}$ if and only if k is even (excluding the cases $l=k$ and $l=0$); the restriction of the fibration over the circle representing a generator of $\pi_1(G_{k,l})$ is always trivial. Hence, in particular, it follows that if n is odd, then the manifold $\Lambda_n^{k,l}$ is orientable if and only if k and l are both even, i.e. when the fiber and the base of the fibration are orientable.

The mapping $i_n^{k,l}$ determines a homology class of the manifold Λ_n (the image of the fundamental class $[\widetilde{\Lambda}_n^{k,l}]$)—with integer coefficients if n is odd and k and l are even, and with coefficients in the group Z_2 for arbitrary n, k, l. If k is even and n and l are odd, then the manifold $\widetilde{\Lambda}_n^{k,l}$ is nonorientable, and its one-dimensional Stiefel class is equal to the image of the generator of the group $H^1(\Lambda_n; Z_2)$ under the homomorphism $(i_n^{k,l})^*$. Therefore in this case the mapping $(i_n^{k,l})_*$ determines a homology class with coefficients in Z_T, the only nontrivial local system of groups isomorphic to Z with base Λ_n. Finally, if $n=k$ is odd and l is also odd, then although the manifold $\widetilde{\Lambda}_n^{k,l}$ is nonorientable, the open manifold $\widetilde{\Lambda}_n^{k,l} \setminus i^{-1}(\Lambda_n^{k,l+1})$ is orientable (the codimension of the singularities of the mapping i is equal to 1). This permits one to regard $\Lambda_n^{n,l}$, for odd l, as an integer cycle. If l is even, then $\Lambda_n^{n,l}$ is a cycle with coefficients in Z_T. Applying Poincaré duality to the homology classes defined by the cycles A_k, B_k, Λ_n^k, we obtain, for every odd n, the cohomology classes respectively

$$
\begin{aligned}
&a_k \in H^k(\Lambda_n; Z_2), && k = 1, 2, \dots, n;\\
&b_k \in \begin{cases} H^{2k+1}(\Lambda_n; Z), \\ H^{2k+1}(\Lambda_n; Z_T), \end{cases} && \begin{matrix} k = 2, 4, \dots, n-1; \\ k = 1, 3, \dots, n-2; \end{matrix}\\
&\lambda_n^k \in \begin{cases} H^{(k(k+1))/2}(\Lambda_n; Z), \\ H^{(k(k+1))/2}(\Lambda_n; Z_T), \end{cases} && \begin{matrix} k = 1, 3, \dots, n; \\ k = 2, 4, \dots, n-1, \end{matrix}
\end{aligned}
$$

with $\rho_2 \lambda_n^1 = a_1$ (ρ_2 is reduction modulo 2).

The same classes can also be defined in the cohomology of Λ_n for even n. This can be done either geometrically (only then the nonorientability, rather than the "coorientability," of the set $\Lambda_n^{k,l}$ will be relevant), or by considering the natural embeddings $\Lambda_{n-1} \subset \Lambda_n \subset \Lambda_{n+1}$. We shall not dwell on this in detail; in the subsequent formulations of all the theorems we regard n as arbitrary, and in the proofs as odd.

Since the space Λ_n is classifying for Lagrangian bundles, in the cohomology of the base X of any Lagrangian bundle ξ there are defined characteristic classes $a_k(\xi) \in H^k(X; Z_2)$, $b_k(\xi) \in H^{2k+1}(X; Z$

or Z_T) and $\lambda_n^k(\xi) \in H^{(k(k+1))/2}(X; Z$ or $Z_T)$. Here Z_T is the local system of groups isomorphic to Z with base X, determined by the bundle ξ.

The classes $a_k(\xi)$ coincide with the Stiefel classes of the bundle ξ (and thus do not depend on the choice of trivialization of the complexification of the bundle ξ). The classes $\lambda_n^k(\xi)$, by definition, are the Maslov-Arnol' d classes of the Lagrangian bundle ξ. Information on the classes $b_k(\xi)$ is contained in Theorem 4 below.

§ 4. **Formulation of the results.**

Theorem 1. $\rho_2 \lambda_n^k = a_1 \cdots a_k = W^1 \cdots W^n$ $(k = 1, 2, \dots, n)$.

Corollary. The Maslov-Arnol' d classes, reduced modulo 2, do not depend on the choice of trivialization of the complexification.

Theorem 2. $\lambda_n^{2s+1} = \lambda_n^1 b_2 b_4 \cdots b_{2s}$ $(s = 1, 2, \dots, (n-1)/2)$.

Theorem 3. $\lambda_n^{2s} = b_1 b_3 \cdots b_{2s-1}$ $(s = 1, 2, \dots, (n-1)/2)$.

Corollary. If any one of the Maslov-Arnol' d classes is equal to zero, then all subsequent ones are equal to zero.

Theorem 4. The ring of weak integral cohomology of the manifold Λ_n for odd n is an exterior algebra with generators $\lambda_n^1, b_2, b_4, \dots, b_{n-1}$.

The last theorem means that the classes λ_n^1, b_k coincide with the generators of the cohomology ring of the homogeneous space $U(n)/O(n)$, found by A. Borel ([3]).

Since Λ_n is the classifying space of Lagrangian bundles, the relations constituting Theorems 1-3 also hold for the characteristic classes $a_k(\xi), b_k(\xi), \lambda_n^k(\xi)$ of any Lagrangian bundle ξ. The proofs of Theorems 1-3 are analogous. In § 5 we shall prove only Theorem 1. § 6 is devoted to the study of the integral cohomology of the space Λ_n. In it, in particular, Theorem 4 is proved.

§ 5. **Intersections.** We shall prove the equality $\rho_2\lambda_n^k = a_k(\rho_2\lambda_n^{k-1})$. By definition, the class $D(\rho_2\lambda_n^{k-1})$ is the image of the fundamental class mod 2 of the manifold $\widetilde{\Lambda}_n^{n,k-1}$ under the mapping $i_n^{n,k-1}$. Fix a small number $\varepsilon > 0$. Denote by $\widetilde{A}_k^{(\varepsilon)}$ the set of pairs (Π, l), where $\Pi \subset E$ is a Lagrangian plane, and l is a line lying in the intersection of Π with the linear span of the vectors $p_1\cos\varepsilon + q_1\sin\varepsilon, \dots, p_{n-k+1}\cos\varepsilon + q_{n-k+1}\sin\varepsilon$. Obviously,

$$\widetilde{A}_k^{(\varepsilon)} \approx \widetilde{\Lambda}_n^{n-k+1,1},$$

and the mapping $i' : \widetilde{A}_k^{(\varepsilon)} \to \Lambda_n$, where $i'(\Pi, l) = \Pi$, is homotopic to $i_n^{n-k+1,1}$. Therefore the image under the mapping i' of the fundamental class mod 2 of the manifold $\widetilde{A}_k^{(\varepsilon)}$ is $D(a_k)$. It turns out that the mapping

$$j = i_n^{n,k-1} \times i' : \widetilde{\Lambda}_n^{n,k-1} \times \widetilde{A}_k^{(\varepsilon)} \to \Lambda_n \times \Lambda_n$$

is transversal-regular with respect to the diagonal $\Delta(\Lambda_n) \subset \Lambda_n \times \Lambda_n$. The complete preimage $j^{-1}(\Delta(\Lambda_n))$ is precisely the manifold whose image of the fundamental class gives, in the sense of Poincaré duality, the product $a_k(\rho_2(\lambda_n^{k-1}))$. Denote by $\widetilde{\Lambda}_n^k$ the manifold whose points are triples (Π, Π', l), where $\Pi \subset E$ is a Lagrangian plane, $\Pi' \subset \Pi \cap P_n$ is a k-dimensional plane, and $l \subset \Pi' \cap P_{n-k+1}$ is a line.

Consider the mapping

$$\eta : \widetilde{\Lambda}_n^k \to \widetilde{\Lambda}_n^{n,k-1} \times \widetilde{A}_k^{(\varepsilon)},$$

where

$$\eta(\Pi, \Pi', l) = ((\varphi_\varepsilon\Pi, \Pi'/l), (\varphi_\varepsilon\Pi, \varphi_\varepsilon l)).$$

Here Π'/l is the orthogonal complement of the line l in the plane Π', $\varphi_\varepsilon : E \to E$ is the transformation rotating the plane $(l, J(l))$ through the angle ε (so that $(\varphi_\varepsilon\Pi, \varphi_\varepsilon l) \in \widetilde{A}_k^{(\varepsilon)}$) and equal to the identity on the orthogonal complement of this plane. It is easy to see that this mapping is a diffeomorphism from $\widetilde{\Lambda}_n^k$ onto $j^{-1}(\Delta(\Lambda_n))$. The composition

$$\widetilde{\Lambda}_n^k \xrightarrow{\eta} \widetilde{\Lambda}_n^{n,k-1} \times A^{(\varepsilon)} \to \Delta(\Lambda_n) \subset \Lambda_n \times \Lambda_n$$

is homotopic to the mapping taking (Π, Π', l) to Π. The latter is the composition of the mapping

$$\widetilde{\Lambda}_n^k \to \Lambda_n^{n,k} \quad ((\Pi, \Pi', l) \to (\Pi, \Pi'))$$

of degree 1 and $i_n^{n,k}$. Hence the assertion being proved follows.

§ 6. The rational cohomology of the space Λ_n for odd n can easily be found from the Cartan-Serre theorem. The ring $H^*(\Lambda_n; Q)$ for odd n is the exterior algebra on generators $\beta_k \in H^{4k-3}(\Lambda_n; Q)$, $k = 1, \dots, (n-1)/2$.

Consider in the space Λ_n the filtration

$$* = \Lambda_n^n \subset \dots \subset \Lambda_n^1 \subset \Lambda_n^0 = \Lambda_n.$$

Construct, with respect to this filtration, the (integral) spectral sequence of cohomology groups. The term $E_1^{p,q}$ of this spectral sequence is

$$H^{p+q}(\Lambda_n^{n-p+1}, \Lambda_n^{n-p});$$

the term E_∞ is associated with $H^*(\Lambda_n)$. It can be shown that this spectral sequence is multiplicative. The difference $\Lambda_n^p \setminus \Lambda_n^{p-1}$ is the space of a vector bundle with base $G_{n,p}$ and fiber of dimension $((n-p)(n-p+1))/2$, and this bundle is orientable if p is odd and nonorientable if p is even. Therefore

$$E_1^p = \sum_q E_1^{p,q}$$

is nothing other than the full cohomology group of the Thom space of this bundle, i.e.

$$E_1^{p,q} = H^{p+q-(p(p+1))/2}(G_{n,n-p}; Z)$$

for even p, and

$$E_1^{p,q} = H^{p+q-(p(p+1))/2}(G_{n,n-p}; Z_T)$$

for odd p, where Z_T is the only possible nontrivial local system over $G_{n,n-p}$ of groups isomorphic to Z. In particular, the groups $E_1^{p,(p(p-1))/2}$ for odd $p \le n$ are isomorphic to Z; all the groups $E_1^{1,q}$ are finite, except for $E_1^{1,0} = Z$; among the groups $E_1^{2,q}$, only the groups $E_1^{2,3+4s}$, $s = 0, \dots, (n-3)/2$, are infinite, each of them being isomorphic to the direct sum of the group Z and a finite group.

A simple count of ranks shows that the analogous spectral sequence with coefficients in the rational numbers is trivial. It follows that the elements of the group $E_1^{p,(p(p-1))/2}$ are not images of any differentials (by dimensional considerations, the cycles of all differentials are elements of this group). Since

$$E_1^{r,(p(p+1))/2-r} = 0$$

for $r > p$, the group $E_\infty^{p,(p(p-1))/2}$ is a subgroup of the group

$$H^{(p(p-1))/2}(\Lambda_n).$$

This subgroup is characterized by the fact that its elements go to zero under the homomorphism induced by the inclusion

$$\Lambda_n^{n-p-1} \subset \Lambda_n.$$

In particular, the class λ_n^p has this property. Moreover, the value of this class on the cycle Λ_{n-p} is equal to 1; consequently, it is not divisible by any integer and hence is a generator of this subgroup.

Thus the classes

$$\lambda_n^p = \lambda_n^1 b_2 \dots b_{p-1} \in H^*(\Lambda_n)$$

are indivisible and have infinite order. Hence, together with the computation above of the rational cohomology of the space Λ_n, Theorem 4 follows.

Let us note that the generators $\lambda_n^1, b_2, \dots, b_{n-1}$ of the ring of weak cohomology of Λ_n correspond in E_1 to the generators of the free parts of the groups

$$E_1^{1,0},\ E_1^{2,3},\ \dots,\ E_1^{2,2n-3}.$$

In conclusion we note that all the results of this note can be obtained solely from the spectral sequence of § 6, without invoking any geometry.

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3. A. Borel, in the collection *Fiber Spaces*, IL, 1958, p. 163.

Note: Figure translations are in progress. See original paper for figures.