

# APPROXIMATION OF DIFFERENTIABLE FUNCTIONS BY FOURIER-HERMITE SUMS

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## APPROXIMATION OF DIFFERENTIABLE FUNCTIONS BY FOURIER-HERMITE SUMS

*(Presented by Academician I. N. Vekua on 29 II 1968)*

Let  $H_n(x)$  be the Hermite polynomials, orthogonal on  $(-\infty, \infty)$  with weight  $e^{-x^2}$  and normalized by the condition

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x) dx = 1$$

(the leading coefficient of  $H_n(x)$  is assumed positive); let  $W^r H_\omega^0$  be the class of functions having on the whole axis a continuous derivative of order  $r = 0, 1, \dots$ , whose modulus of continuity satisfies  $\omega(f^{(r)}, \delta) \leq \omega(\delta)$ , where  $\omega(\delta)$  is a prescribed true majorant of moduli of continuity and

$$f^{(r)}(0) = 0; \quad s_{n-1}(f; x) = \sum_{k=0}^{n-1} c_k(f) H_k(x), \quad c_k(f) = \int_{-\infty}^{\infty} e^{-x^2} f(x) H_k(x) dx;$$

$$\Delta_n(f; x) = f(x) - s_{n-1}(f; x); \quad c_\omega^{(m)} = \sup_{f \in H_\omega(2\pi)} \int_{-\pi}^{\pi} f(t) \cos nt dt,$$

where  $H_\omega(2\pi)$  is the class of continuous  $2\pi$ -periodic functions  $f(x)$  for which  $\omega(f, \delta) \leq \omega(\delta)$ ;  $p = \sqrt{2n}$ ,  $m = [p]$ ,  $q = \sqrt{2k}$ .

In <sup>(1)</sup> an asymptotic formula was obtained for

$$\sup_{f \in W^0 H_\omega^0} |\Delta_n(f; x)|$$

with a remainder term uniformly bounded for  $x$  in a finite interval. The purpose of our work is to extend this result to the class  $W^r H_\omega^0$  ( $r = 1, 2, \dots$ ).

**Theorem.** For  $r = 1, 2, \dots$  and any  $x \in (-\infty, \infty)$ ,

$$\sup |\Delta_n(f; x)| = \frac{\ln m}{\pi^2 m^r} c_\omega^{(m)} + \rho_m(x),$$

where

$$\rho_m(x) = e^{x^2/2} (1 + |x|^{11/2}) O\left(\frac{1}{m^r} \omega\left(\frac{1}{m}\right)\right),$$

and the constant contained in  $O$  depends only on  $r$ .

**Remark.** In the proof of the theorem, the usual method, based on the summation of sums of the form  $\sum_{k=n}^N \sin kx$ , which works in the theory of Fourier series <sup>(2)</sup>, Fourier-Jacobi series <sup>(3)</sup>, does not apply to Fourier-Hermite series, since here sums of the form  $\sum_{k=n}^N \sin \sqrt{k}x$  arise. We overcome this difficulty by means of Lemma 4 (see below).

The proof of the theorem rests on the following lemmas.

**Lemma 1.** Let

$$x_k = \frac{n - 2[n/2] + 2k}{\sqrt{2(n+1)}} \frac{\pi}{2} \quad (k = 1, 2, \dots, n).$$

Then

$$\sum_{k=1}^n e^{-\theta x_k^2} = O(p).$$

Here  $0 < \theta < 1$ .

**Lemma 2.** If  $f(t) \in H_\omega^0$ , then for  $|h| \leq \delta \leq 1$ , for  $F(t) = e^{-t^2/2} f(t)$ , one has

$$F(t+h) - F(t) = e^{-\varepsilon t^2} O(\omega(\delta)),$$

$$F(t+h) \sin a(t+h) - F(t) \sin at = e^{-\varepsilon t^2} O(a\omega(\delta)),$$

where  $a$  is an arbitrary number and  $\varepsilon > 0$ .

**Lemma 3.** Let

$$F(t) = \frac{1}{t-x} \left( e^{(x^2-t^2)/2} - 1 \right).$$

Then for  $|t-x| \leq 1$  one has

$$F(t) = O(1 + |x|^3) e^{|x|}$$

and, moreover,

$$\int_{c/p \leq |t-x| \leq 1} F(t) [f(t) - f(x)] \sin p(t-x) dt = p^r \rho_n(x),$$

if  $f(t) \in H_\omega^c$ .

**Lemma 4.** The equality

$$\sum_{k=n}^{\infty} \frac{\cos(qt + r\pi/2)}{q^{r+1}} = \int_p^{\infty} \frac{\cos(tz + r\pi/2)}{z^r} dz + O(1+t^2)p^{-r-1}$$

is valid.

**Lemma 5.** If  $f(t) \in H_{\omega}^0$ , then

$$c_n(f) = \sqrt{\frac{2}{\pi p}} \int_{-\infty}^{\infty} e^{-x^2/2} f(x) \cos\left(px - \frac{n\pi}{2}\right) dx + O(p^{-3/2}\omega(p^{-1})).$$

**Lemma 6.** If  $f(t) \in H_{\omega}^0$ , then

$$\int_{c/p}^1 f(t) \frac{\cos pt}{t^2} dt = O(p\omega(p^{-1})), \quad \int_p^{\infty} \frac{dz}{z^{r+2}} \left( \int_{c/p}^1 f(t) \frac{\cos zt}{t^2} dt \right) = O(p^{-r}\omega(p^{-1})).$$

These formulas remain valid if  $\cos t$  is replaced in them by  $\sin t$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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