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T. A. GERMOGENOVA

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Abstract

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T. A. GERMOGENOVA

ON THE CONVERGENCE OF SOME APPROXIMATE METHODS FOR SOLVING THE TRANSPORT EQUATION

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Recently, in the numerical solution of the transport equation, the use of approximate methods has been characteristic, in which the iterative process is combined with various devices for accelerating convergence. A number of authors ([1−3]) construct these methods on the basis of the use of variational principles proposed in ([4]) for the one-velocity problem with an even indicatrix, and in ([5]) for more general symmetrizable problems. The theory of the functional properties of the transport equation developed in ([4,5]) makes it possible to understand the mechanism of convergence of methods of this type, and also to obtain a justification of a number of approximate algorithms in problems not reducible to self-adjoint form.

We shall first consider the symmetrizable case—the general one-velocity problem and the energy problem with the condition of detailed balance for a bounded domain $G \subset R_3$ with piecewise-smooth boundary Γ and a finite interval of velocities v. In this case the first boundary-value problem for the transport equation, i.e., the problem with prescribed on Γ, for $\mathbf{vn} < 0$, values of the intensity of the radiation entering G, is equivalent to the following self-adjoint problem ([5]):

$$(\mathcal{L}_0 - S_1)U(\mathbf{r}, \mathbf{v}) = F(\mathbf{r}, \mathbf{v}) \quad \text{in } G \times V;$$

on Γ:

$$U = (E - S_2)^{-1} \left(\frac{\mathbf{v}\vec{\nabla}}{h} U - G \right) \quad \text{for } \mathbf{vn} < 0,$$

$$U = -(E - S_2)^{-1} \left(\frac{\mathbf{v}\vec{\nabla}}{h} U - G \right) \quad \text{for } \mathbf{vn} > 0, \tag{1}$$

where S_1 and S_2 are integral operators of the form $(i = 1, 2)$

$$S_iU(\mathbf{r},\mathbf{v}) = \int_V s_i(\mathbf{r};\mathbf{v}',\mathbf{v})U(\mathbf{r},\mathbf{v}')\,dv',$$

with

$$s_i(\mathbf{r};\mathbf{v}',\mathbf{v}) = (-1)^i s_i(\mathbf{r};-\mathbf{v}',\mathbf{v}) = (-1)^i s_i(\mathbf{r};\mathbf{v}',-\mathbf{v}),$$

and

$$\mathcal{L}_0U = -\frac{\mathbf{v}\vec{\nabla}}{h}(E-S_2)^{-1}\frac{\mathbf{v}\vec{\nabla}}{h}U;$$

$\mathbf{n}$ is the outward normal to the boundary Γ; F, G, h are prescribed functions.

Under a number of natural restrictions on F, G, h, s_i, problem (1) has a generalized solution, unique for a nonmultiplying medium, and finding it in this case, by virtue of the positive definiteness of $\mathcal{L}_0 - S_1$, is equivalent to finding the minimum of the quadratic functional ([6])

$$J[U] = ((\mathcal{L}_0 - S_1)U, U) - 2(F,U). \tag{2}$$

Theorem 1. *Any method of approximate solution of problem (1) based on alternating simple iteration and procedures that allow one, from a certain class W of functions containing the last computed iteration U_n, to choose that function $\widetilde{U}_{n+1/2}$ which gives $J[U]$ the least value in this class, will give a process converging in the norm $L_2(G\times V)$ to the solution of problem (1).*

For the proof it is sufficient to show that the resulting sequence of approximate solutions minimizes the functional $J[U]$.

Let $\overline{U}$ be the exact solution of problem (1), U_{n-1} the approximate one, $U_n = \mathcal{L}_0^{-1}[S_1U_{n-1}+F]$, $\varepsilon_k = U_k - \overline{U}$. Then

$$J[U_k] = J[\overline{U}] + ((\mathcal{L}_0 - S_1)\varepsilon_k, \varepsilon_k) \qquad (k = n-1, n)$$

and, if we put $\varepsilon_{k-1} = \sum_i a_i\Phi_i + \chi$, where Φ_i are the eigenfunctions corresponding to the homogeneous problem,

$$\Phi_i = \lambda_i \mathcal{L}_0^{-1} S_1 \Phi_i,$$

then, since $\lambda_i > 1$, $(S_1\Phi_i, \Phi_l) = \delta_{il}$, $(S_1\Phi_i, \chi) = 0$ ([4,5]), we shall have

$$J[U_{n-1}] = J[\overline{U}] + \sum_i a_i^2(\lambda_i - 1) + ((\mathcal{L}_0 - S_1)\chi, \chi) >$$

$$> J[\overline{U}] + \sum_i a_i^2 \frac{\lambda_i - 1}{\lambda_i^2} = J[U_n].$$

Hence follows the convergence of the sequence of approximate solutions in the energy norm $|\Phi|^2 = ((\mathcal{L}_0 - S_1)\Phi, \Phi)$, and since $((\mathcal{L}_0 - S_1)\Phi, \Phi) \geq \gamma_1(\Phi, \Phi)$, $\gamma_1 > 0$, also in L_2.

Recently it has been shown ([3,5,7]) that modifications of methods of the type under consideration include such effective methods as the quasidiffusion method ([8]), the method of mean fluxes ([1,5]), and KP-methods ([2,7]). If the approximation $U_{n+1/2}$ is sought in the form $U_{n+1/2} = U_n + w_n$, then, putting

$$w_n = \sum_i b_i \Phi_i + v,$$

we shall have

$$J[U_{n+1/2}] = J[\overline{U}] + \sum_i \left(\frac{a_i}{\lambda_i} + b_i\right)^2 (\lambda_i - 1) + ((\mathcal{L}_0 - S_1)v, v).$$

Since in the simple iterative process the slow convergence of the iterations is connected with the closeness of the first characteristic numbers $\lambda_1, \lambda_2, \dots, \lambda_N$ to 1, the effectiveness of the correction w_n is determined by the possibility of setting to 0 the coefficients $a_i/\lambda_i + b_i$ for $i = 1, 2, \dots, N$, i.e. by how close the functions of the class W are to the first eigenfunctions of the problem.

It is known that the sequence of approximate solutions $U_0, U_1, \dots, \dots, U_n$ of problem (1) carries information about the eigenfunctions of the problem. Therefore nonlinear acceleration methods are of considerable interest, in which the choice of $W^{(n)}$ is determined by the functions $U_0, U_1, \dots, U_n$.

Let w_n be sought in the form of a sum of products

$$w_n = \sum_{k=1}^{I_n} m_k(\mathbf{r})\, U_k^{(n)}(\mathbf{r}, \mathbf{v}), \tag{3}$$

where $U_k^{(n)}$ are certain weight functions constructed from $U_0, U_1, \dots, U_n$. For $I_n = 1$, $U_1^{(n)} = U_n$, we obtain the method of mean fluxes ([1]): $U_{n+1/2} = N(r)U_n$, $N = m_1 + 1$. For $I = 2$, $U_1^{(n)} = U_n$, $U_2^{(n)} = U_{n-1}$, $m_{1,2} = \text{const}$, we arrive at a modification of the method of normalized extrapolations proposed by V. N. Morozov in ([9]), where for m_1, m_2 expressions were used whose basis is Lyusternik' s extrapolation formula, and normalization of the approximate solution on the basis of the balance relation. Minimization of J in the class of functions

$$\vartheta = \widetilde{m}_2[U_n + \widetilde{m}_1(U_n - U_{n-1})]$$

successively with respect to the parameters $\widetilde{m}_1$ and $\widetilde{m}_2$ leads to expressions for m_1 and m_2 differing from those obtained in ([9]) only by the weights U_n, U_{n-1} in the integrals.

For approximations of the general type (3), the following assertions can be proved.

Theorem 2. If $U_k^{(n)}$ satisfy the conditions

$$\sum_{k,k'} \int_G dr\, m_k(\mathbf{r}) m_{k'}(\mathbf{r}) \int_V U_k^{(n)}(\mathbf{r},\mathbf{v}) U_{k'}^{(n)}(\mathbf{r},\mathbf{v})\, d\mathbf{v} \geq \gamma_2 \sum_k \int_G m_k^2\, dr,$$

where $\gamma_2 > 0$, for arbitrary $m_k, m_{k'} \in L_2(G)$, then:

1) the variational problem for $\{m_k\}_1^{I_n}$ has a unique solution in the space M, whose elements are sequences $\{m_1, \dots, m_{I_n}\}$ of functions $m_i \in L_2$ possessing all generalized derivatives $dm_k/dx_j \in L_2$ $(j = 1, 2, 3)$;

2) the boundary-value problem for $\{m_k\}_1^{I_n}$ is of elliptic type with the oblique derivative m_k equal to zero on Γ.

The basis of the proof is the study of the variational problem that arises for $\{m_i\}$.

Numerous experimental computations for the transport equation by the method of mean fluxes were carried out by T. A. Sushkevich.

Analysis of the results of the computations ([1]) makes it possible to regard this method as very effective and little sensitive to changes in the parameters of the problem.

In order to give an idea of the rate of convergence of the method, we present some results of the calculation of a one-velocity problem on the propagation of radiation incident on the surface $x = 0$ of a homogeneous plane layer of thickness 10 mean free paths, with pure scattering (isotropic). To attain an accuracy $\varepsilon = 10^{-5}$, about 400 simple iterations are required (time on a BESM-4 $t_1 = 90$ min.). By the method of mean fluxes the same results are obtained in 8 iterations ($t_2 = 2$ min.). Some idea of the character of the convergence can be obtained from Fig. 1. The ratio t_1/t_2 changes little when strongly anisotropic scattering is introduced, when there is inhomogeneity in the layer, and when one passes to spherical geometry.

Fig. 1. Results of the calculation of the radiation density

$$\int_{-1}^{+1} U(x,\mu)\, d\mu$$

by the method of mean fluxes in the first approximations. The solid line corresponds to iterative steps, the dashed line to variational ones. (In the computation the non-self-adjoint form of the transport equation was used.)

As examples of non-symmetrizable problems, problems for multiplying media were considered, where the detailed-balance relation does not hold, and the II boundary-value problem (i.e., the problem with reflection conditions on the boundary Γ) ([5]). In these cases the boundary-value problem is reduced to the functional equation

$$AU = A_0U + TU = f,$$

where A_0 is the adjoint operator, and T is such that $A_0^{-1}T$ is completely continuous in $L_2(G \times V)$, which makes it possible, in accordance with ([6]), to verify the convergence of methods of the Bubnov–Galerkin type.

We also note that the construction and analysis of methods of the iteration-variational type can be carried out in an analogous way for the solution of homogeneous problems, using the corresponding variational principles ([4,5]).

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Note: Figure translations are in progress. See original paper for figures.