A MIXED PROBLEM WITH DISCONTINUOUS BOUNDARY CONDITIONS FOR THE WAVE EQUATION

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Abstract

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A MIXED PROBLEM WITH DISCONTINUOUS BOUNDARY CONDITIONS FOR THE WAVE EQUATION

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The paper considers a mixed problem with discontinuous boundary conditions for the wave equation

$$\frac{\partial^2 v}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} = f(t,x), \qquad t > 0, \quad x_1 > 0, \quad x = (x_1, \dots, x_n)$$

with initial conditions

$$v(+0,x) = \varphi_0(x), \qquad \frac{\partial v}{\partial t}(+0,x) = \varphi_1(x), \quad x_1 > 0$$

and boundary conditions

$$v(t,+0,x') = g_0(t,x'), \quad x_2 > 0,\ t > 0, \quad x' = (x_2, \dots, x_n),$$

$$\frac{\partial v}{\partial x_1}(t,+0,x') = g_1(t,x'), \quad x_2 < 0,\ t > 0, \quad \text{where } x' = (x_2, \dots, x_n).$$

Solving the auxiliary Cauchy problem and making the substitution $v(x,t) = u(x,t)e^{at}$, $a > 0$, we reduce this problem to the form

$$A(i\partial/\partial t,\ i\partial/\partial x) \equiv (\partial/\partial t + a)^2 u - \Delta u = 0, \qquad t > 0,\ x_1 > 0; \tag{1}$$

$$u(+0,x) = 0, \qquad \frac{\partial u}{\partial t}(+0,x) = 0, \quad x_1 > 0; \tag{2}$$

$$u(t,+0,x') = h_0(t,x'), \qquad x_2 > 0,\ t > 0,$$

$$\frac{\partial u}{\partial x_1}(t, +0, x') = h_1(t, x'), \qquad x_2 < 0, \quad t > 0. \tag{3}$$

Two questions are investigated: 1) the precise formulation of the boundary-value problem (1)–(3); 2) the smoothness of the solution.

The first question arises in connection with the fact that we cannot expect the existence of a classical solution of this problem for smooth h_0 and h_1 satisfying local compatibility conditions (of finite number) on the manifold of discontinuity.

In formulating the problem, the projection operators in L_2, Π^+ and Π^-, described in (1), are used.

We introduce the necessary function spaces. Let $u(t, x) \in L_2(R^{n+1})$. Its Fourier transform is

$$Fu \equiv u(\tau, \xi) = \int e^{it\tau + i(x\xi)} u(t, x)\, dt\, dx.$$

To the operator $A(i\partial/\partial t,\ i\partial/\partial x)$ from (1) there corresponds, in Fourier images, the operator of multiplication by

$$A(\tau, \xi) = (-i\tau + a)^2 + |\xi|^2 = (\xi_1 - \lambda)(\xi_1 + \lambda).$$

Here

$$\lambda \equiv \lambda(\tau, \xi') = \sqrt{(\tau + ia)^2 - |\xi'|^2} = \sqrt{(\tau + ia)^2 - |\xi''|^2 - \xi_2^2}, \quad \xi'' = (\xi_3, \dots, \xi_n),$$

with the condition

$$\lambda(0, \xi') = i\sqrt{a^2 + |\xi'|^2}.$$

The function $\lambda(s, \xi')$ is analytic in the complex plane $s = \tau + i\sigma$ with a cut from the point $s_1 = |\xi'| - ia$ to the point $s_2 = -|\xi'| - ia$ in the lower half-plane. Let

$$\varkappa(\tau, \xi'') = \lambda(\tau, 0, \xi'').$$

Then

$$\lambda(\tau, \xi') = \sqrt{(\varkappa - \xi_2)(\varkappa + \xi_2)}.$$

The function $\lambda(\tau, z_2, \xi'')$ is analytic in the complex plane

planes $z_2 = \xi_2 + i\eta_2$ with cuts from the points $\pm\varkappa$ to ∞, respectively, in the upper and lower half-planes. In the same plane with such cuts we consider the functions $\eta_\pm(\tau, z_2, \xi'') = \sqrt{\varkappa \pm z_2}$, choosing the branches with the conditions $\eta_+ \to \sqrt{\xi_2}$ as $\xi_2 \to +\infty$, $\eta_- \to i\sqrt{\xi_2}$ as $\xi_2 \to +\infty$. We have $\lambda(\tau, \xi') = \sqrt{\varkappa + \xi_2}\sqrt{\varkappa - \xi_2} = \eta_+\eta_-$. Let us also note that if $\lambda(\tau, \xi') = \operatorname{Re}\lambda + i\operatorname{Im}\lambda$, then $\operatorname{Im}\lambda(\tau, \xi') \geq a$, $\operatorname{Im}\lambda \to a$ as $|\tau| \to \infty$, and $\operatorname{Im}\lambda = O(\sqrt{a^2 + |\xi'|^2})$ as $|\xi'| \to \infty$.

Definition 1.

$$V_s(R^{n+1}) = \Big\{ u(t, x) \in L_2(R^{n+1}) : |u|^2_{V_s(R^{n+1})} =$$

$$= \int \frac{|A(\tau,\xi)|^2 |\varkappa|^{1/2} (\operatorname{Im}\varkappa)^{1/2}}{|\eta_+|^2 \operatorname{Im}\lambda} \left| \frac{A\varkappa \operatorname{Im}\varkappa}{\lambda \operatorname{Im}\lambda} \right|^{2(s-1)} |u(\tau,\xi)|^2 \, d\tau d\xi < \infty \Bigg\},$$

$s \geq 1$ an integer.

It can be proved that

$$V_s(R^{n+1}) \subset W_2^{(1,1,s,s)_{(x_1,x_2,x'',t)}}(R^{n+1}) =$$

$$= \Big\{ u(t,x) \in L_2(R^{n+1}) : \int (1 + |\xi_1|^2 + |\xi_2|^2 + |\xi''|^{2s} + |\tau|^{2s}) |u(\tau,\xi)|^2 \, d\tau d\xi < \infty \Big\}.$$

Definition 2.

$$X_s^0(R^n) = \{ v(t,x) \in L_2(R^n) :$$

$$|v|^2_{X_s^0(R_n)} = \int |\eta_-|^2 |\varkappa|^{3/2} (\operatorname{Im}\varkappa)^{1/2} |\varkappa \operatorname{Im}\varkappa|^{2(s-1)} |v(\tau,\xi')|^2 \, d\tau d\xi' < \infty \Big\}.$$

It can be proved that

$$X_s^0(R^n) \subset W_2^{(1/2,\ s-1/4,\ s-1/4)_{(x_2,x'',t)}}(R^n).$$

Definition 3.

$$X_s^1(R^n) = \Big\{ v(t,x') \in W_2^{(-1/2)}(R^n) :$$

$$|v|^2_{X_s^1(R^n)} = \int \frac{|\varkappa|^{5/2} (\operatorname{Im}\varkappa)^{1/2}}{|\eta_+|^2} |\varkappa \operatorname{Im}\varkappa|^{2(s-1)} |v(\tau,\xi')|^2 \, d\tau d\xi' < \infty \Big\}.$$

It can be proved that

$$X_s^1(R^n) \subset W_2^{(-1/2),(s-3/4,\ s-3/4)}{}_{(x_2),(x'',t)}(R^n) =$$

$$= \Big\{ v(t,x') \in W_2^{(-1/2)}(R^n) : \int \frac{(1 + \tau^2 + |\xi''|^2)^{s-3/4}}{(1 + \xi_2^2)^{1/2}} |v(\tau,\xi')|^2 \, d\tau d\xi' < \infty \Big\}.$$

Let us note that $X_s^0(R^n)$ and $X_s^1(R^n)$ are the natural spaces of traces of the functions $u(t, x_1, x')$ and $\dfrac{\partial u}{\partial x_1}(t, x_1, x')$ for $x_1 = \mathrm{const}$, where $u(t, x) \in V_s(R^{n+1})$.

Let $R_+^{n+1} = \{(t, x) : x_1 > 0\}$.

Definition 4.

$$V_s^+(R_+^{n+1}) = \Big\{u(t, x) \in L_2(R_+^{n+1}) :$$

$$\exists u \text{ and } \hat{u}(t, x) \in V_s(R^{n+1}) : \ u(t, x) = \hat{u}(t, x), \ x_1 > 0;$$

$$|u|_{V_s^+(R_+^{n+1})} = \inf_{\hat{u}} |\hat{u}(t, x)|_{V_s(R^{n+1})}\Big\}.$$

We now proceed to the formulation of the problem in terms of the spaces introduced. Let $h_0(t, x')$, $h_1(t, x')$ from (3) admit extensions $\widehat{h}_0(t, x')$ and $\widehat{h}_1(t, x')$ for all x_2 and t in the classes $X_s^0(R^n)$ and $X_s^1(R^n)$, respectively. We shall assume that these functions are extended by zero for $t < 0$ (this imposes on them certain compatibility conditions with zero at $t = 0$). Then the Fourier transforms $\widehat{h}_0(\tau, \xi')$ and $\widehat{h}_1(\tau, \xi')$ will have analytic continuations $H_0(s, \xi')$ and $H_1(s, \xi')$ in $\operatorname{Im} s > 0$, $s = \tau + i\sigma$, and

$$|H_0(s, \xi')| \leq c(\xi'), \qquad |H_1(s, \xi')| \leq c(\xi'). \tag{4}$$

Extend the functions from $V_s^+(R_+^{n+1})$ by zero for $x_1 < 0$, denoting these extensions by $u^+(t, x)$. We denote by V_s^+ the class of their Fourier transforms $u^+(\tau, \xi)$. These functions admit a bounded analytic continuation into the half-plane $\operatorname{Im} z_1 > 0$, where $z_1 = \xi_1 + i\eta_1$. Let $v_-(t, x') \in X_s^0(R^n)$, $w_+(t, x') \in X_s^1(R^n)$, and be equal to zero for $x_2 < 0$ and $x_2 > 0$, respectively. Then their Fourier transforms $v_-(\tau, \xi')$, $w_+(\tau, \xi')$ admit analytic continuations $V_-(\tau, z_2, \xi'')$, $W_+(\tau, z_2, \xi'')$ into the complex half-planes $\operatorname{Im} z_2 < 0$ and $\operatorname{Im} z_2 > 0$, respectively, and

$$|V_-(\tau, z_2, \xi'')| \leq c(\tau, \xi''), \qquad \operatorname{Im} z_2 < 0; \tag{5}$$

$$|W_+(\tau, z_2, \xi'')| \leq c(\tau, \xi'')|z_2|^{1/2}, \qquad \operatorname{Im} z_2 > 0. \tag{6}$$

For the classes of Fourier transforms of functions from $X_s^0(R^n)$, $X_s^1(R^n)$ we retain the same notation.

Formulation of the problem: find $u^+(\tau, \xi)$, $v_-(\tau, \xi')$, $w_+(\tau, \xi')$ such that:

1°. $u^+(\tau, \xi) \in V_s^+$.

2°. $u^+(\tau,\xi)$ admits an analytic continuation into the complex half-plane $\operatorname{Im} s > 0$, $s = \tau + i\sigma$, and $|u^+(s,\xi)| \le c(\xi)$, $\operatorname{Im} s > 0$.

3°. $\Pi^+_{\xi_1} A(\tau,\xi) u^+(\tau,\xi) = 0$.

4°.

$$\frac{1}{2\pi}\int u^+(\tau,\xi)\,d\xi_1 = \hat{h}_0(\tau,\xi') + v_-(\tau,\xi').$$

5°.

$$\frac{1}{2\pi}\int \Pi^+_{\xi_1}(-i\xi_1)u^+(\tau,\xi_1,\xi')\,d\xi_1 = \hat{h}_1(\tau,\xi') + w_+(\tau,\xi').$$

6°. $v_-(\tau,\xi') \in X^0_s(R^n)$ and satisfies (5).

7°. $w_+(\tau,\xi') \in X^1_s(R^n)$ and satisfies (6).

8°. $v_-(\tau,\xi')$, $w_+(\tau,\xi')$ admit analytic continuations $V_-(s,\xi')$, $W_+(s,\xi')$ into the complex half-plane $\operatorname{Im} s > 0$, with

$$|V_-(s,\xi')| \le c(\xi'), \qquad |W_+(s,\xi')| \le c(\xi').$$

Definition. By a generalized solution of problem (1)–(3) we shall mean a function $u(t,x) \in V^+_s(R^{n+1}_+)$ satisfying 1°–5°.

The functions $v_-(t,x')$, $w_+(t,x')$ appear in the formulation of the problem because of the arbitrariness in the continuation of $h_0(t,x')$, $h_1(t,x')$ for $x_2 < 0$, $x_2 > 0$, respectively.

Theorem. The solution of problem 1°–8° exists and is unique in $V^+_s \times X^0_s \times X^1_s$ for arbitrary $\hat{h}_0(t,x') \in X^0_s(R^n)$ and $\hat{h}_1(t,x') \in X^1_s(R^n)$, and the estimate holds

$$|u|_{V^+_s(R^{n+1}_+)} \le c\left\{\left\|\varkappa^{3/4}(\operatorname{Im}\varkappa)^{1/4}(\varkappa\operatorname{Im}\varkappa)^{s-1}\Pi^+_{\xi_2}\eta_-\hat{h}_0(\tau,\xi')\right\|_{L_2(R^n)} + \right.$$

$$\left. + \left\|\varkappa^{3/4}(\operatorname{Im}\varkappa)^{1/4}(\varkappa\operatorname{Im}\varkappa)^{s-1}\Pi^-_{\xi_2}\frac{\hat{h}_1(\tau,\xi')}{\eta_+}\right\|_{L_2(R^n)}\right\}. \tag{7}$$

We note that the norms on the right do not depend on the continuations of $h_0(t,x')$ and $h_1(t,x')$ in the classes $X^0_s(R^n)$ and $X^1_s(R^n)$, but depend only on the functions themselves.

It is proved that the function $u_+(t,x)$ from 1°–8° has the form

$$u^+(\tau,\xi) = \frac{2\left[\Pi^+_{\xi_2}\eta_-(\tau,\xi')\hat{h}_0(\tau,\xi') + i\Pi^-_{\xi_2}\hat{h}_1(\tau,\xi')/\eta_+(\tau,\xi')\right]}{i\eta_-(\tau,\xi')(\xi_1+\lambda(\tau,\xi'))}. \tag{8}$$

Formula (8) shows that, as the domain of definition of the operator $\mathfrak{A}$ described in items 1°–8°, one cannot take a space of the type $W_2^s(R_+^{n+1})$ and the spaces of traces of functions from $W_2^s(R^{n+1})$. The natural domain of definition of this operator is

$$V_s^+(R_+^{n+1}) \times X_s^0(R^n) \times X_s^1(R^n),$$

$s \geq 1$, with nonsmoothness in x_1 and x_2 that cannot be improved. Finally, using the finiteness of the domain of dependence for the solution $u^+(t,x)$ from 1°–8°, one can prove that it will have smoothness of type V_s inside the characteristic "cone," whose "axis" is the manifold of discontinuity of the boundary

conditions $x_1 = 0,\ x_2 = 0$, and will have the smoothness corresponding to an ordinary mixed problem outside it.

As for the "interior smoothness" of the solution, it has been proved that in the region $R^+_{+\delta_1} = \{(t,x) : x_1 \geqslant 0 > 0\}$ the solution has the smoothness of the solution of an ordinary mixed problem. In particular, from the partial hypoellipticity of the hyperbolic equation with respect to x and the proved smoothness properties up to the boundary, it follows that if $h_0(t,x')$ and $h_1(t,x') \in C_0^\infty(R^n)$, then $u^+(t,x) \in C^\infty(R^{n+1})$ (see (3), p. 146, Theorem 4.2.4.). This fact is a consequence of the discontinuity of the boundary conditions occurring on a "time-like" manifold which nowhere intersects the characteristic surfaces.

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Note: Figure translations are in progress. See original paper for figures.