CRAMÉR' S, LINDEBERG' S, AND CHEBYSHEV' S THEOREMS FOR COMPLEX DISTRIBUTIONS

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Abstract

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CRAMÉR' S, LINDEBERG' S, AND CHEBYSHEV'S THEOREMS FOR COMPLEX DISTRIBUTIONS

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A complex continuous function of bounded variation satisfying the conditions $F(-\infty) = 0,\ F(+\infty) = 1$ is called the distribution function $F(x)$ of a random variable ξ taking values in $R = (-\infty, +\infty)$. Denote by $V(F, x)$ the variation of $F(x)$ on $(-\infty, x)$. Many properties of classical (real, nondecreasing) distribution functions and of the corresponding characteristic functions

$$f(t) = \int (\exp itx)\, dF(x)$$

are preserved, in particular the theorem on the one-to-one correspondence between distribution functions and characteristic functions. A sequence of characteristic functions $f_n(t)$ is called convergent to the characteristic function $f(t)$ $(f_n \Rightarrow f)$, if f_n converges to f in the sense of weak convergence over the space K. A sequence of distribution functions F_n converges to the distribution function F in the sense of weak convergence over the space Z if and only if the sequence of the corresponding characteristic functions $f_n \Rightarrow f$. For the definition of the spaces K and Z, see ([1]).

A random variable ξ is called normally distributed, or normal with parameter $a_{2q}^r = (a_1, \dots, a_{2q}, \dots, a_r)$ (q and r are integers, $2q \leq r$), if its characteristic function has the form

$$\varphi(t) = \exp \sum_{p=1}^{r} a_p t^p.$$

The function $\varphi(t)$ will be characteristic if and only if the equation

$$\frac{\partial u}{\partial \tau} = \sum_{p=1}^{r} a_p \left(i \frac{\partial}{\partial x} \right)^p u$$

is parabolic in the sense of G. E. Shilov ([2]), or hyperbolic (when $r = 1$).

Addition theorem. *The sum of independent normal variables is normal. The parameter of the sum is the sum of the parameters.*

The proof is obvious.

Converse theorem (Cramér). *Let the random variables ξ_1 and ξ_2 be independent, and let their distribution functions F_1 and F_2 satisfy the conditions $(i = 1, 2)$:*

$$\text{K1.} \quad V(F_i, R) - V(F_i, x) < A \exp(-\varepsilon x^{\gamma_i}).$$

$$\text{K2.} \quad V(F_i, -x) < A \exp(-\varepsilon x^{\gamma_i})$$

for $x > 0$ and some $\varepsilon > 0$, $\gamma_i > 1$. If the sum $\xi = \xi_1 + \xi_2$ is normal with parameter a_{2q}^r, then ξ_1 and ξ_2 are also normal with parameters $a_{2q_1}^{r_1}$, $a_{2q_2}^{r_2}$, respectively, where $r_1 \le \gamma_1'$, $r_2 \le \gamma_2'$, $\max(q_1, q_2) = q$, where $1/\gamma_i + 1/\gamma_i' = 1$.

Proof. Elementary estimates show that $f_i(t)$ are entire analytic functions of the complex variable t of growth order

$\leqslant \gamma_i'$. Obviously, the $f_i(t)$ have no zeros. By Hadamard' s theorem,

$$f_i(t) = \exp P_i(t),$$

where $P_i(t)$ is a polynomial of degree $r_i \leqslant \gamma_i'$.

Remark. For real symmetric distributions F_1, F_2 and the ordinary normal distribution with parameter $a_2^2 = (0, 1)$, the corresponding converse theorem was proved in ([3]). There, too, an example is constructed showing that the converse theorem becomes false even if, in conditions K1, K2, x^{γ_i} is replaced by $x \ln x$.

Central limit theorem (Lindeberg). Let $\xi_1, \dots, \xi_k, \dots$ be a sequence of independent random variables; $F_1, \dots, F_k, \dots$ and $f_1, \dots, f_k, \dots$ the sequence of the corresponding distribution functions and characteristic functions. Suppose that the following conditions are satisfied:

L1. For every $k = 1, 2, \dots$ and some integer $q \geqslant 1$,

$$\int x \, dF_k(x) = \dots = \int x^{2q-1} \, dF_k(x) = 0; \qquad \int x^{2q} \, dF_k(x) = (-1)^{q+1}(a_k + ib_k).$$

L2. There exists a constant $C < \infty$ such that, for every n,

$$\frac{1}{n} \sum_{k=1}^{n} \int x^{2q} \, dV(F_k, x) \leqslant C.$$

L3. Uniformly in t on every finite interval,

$$\max_{1\leqslant k\leqslant n}\left|f_k\left(\frac{t}{n^{1/2q}}\right)-1\right|\to 0\qquad (n\to\infty).$$

L4. For every $\lambda > 0$, the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_{|x|>\lambda n^{1/2q}} x^{2q}\,dV(F_k,x)=0.$$

L5. There exists the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(a_k+ib_k)=a+ib.$$

Then the distributions of the sums

$$\eta_n=\frac{1}{n^{1/2q}}\sum_{k=1}^{n}\xi_k$$

have a limit if and only if $a > 0$. This limit is the normal distribution with parameter

$$a_{2q}^{2q}=(0,\dots,0,-(a+ib)/(2q)!).$$

Corollary 1. The preceding theorem remains true if condition L3 is replaced by the condition

L6. As $n\to\infty$,

$$\max_{1\leqslant k\leqslant n}\frac{1}{n}\int x^{2q}\,dV(F_k,x)\to 0$$

or by the condition

L7. The limit

$$\lim_{\varepsilon\to 0}\max_{\substack{1\leqslant k\leqslant n\\ 1\leqslant n<\infty}}\frac{1}{n}\int_{|x|<\varepsilon n^{1/2q}} x^{2q}\,dV(F_k,x)=0.$$

The theorem is true if, instead of condition L4, the following condition is satisfied:

L8. There is a $\delta > 0$ such that, as $n\to\infty$,

$$\frac{1}{n^{2q+\delta}} \sum_{k=1}^{n} \int |x|^{2q+\delta}\, dV(F_k, x) \to 0.$$

Corollary 2. Suppose the following condition of uniform boundedness of the variations is satisfied:

L9. There exists a constant $B < \infty$ such that, for every $k = 1, 2, \ldots$,

$$\int dV(F_k, x) \leqslant B.$$

Then the preceding theorem remains true even without condition L3.

Remark. For the case when $\xi_1, \ldots, \xi_k, \ldots$ have real densities and conditions L1, L2, L7, L4 are satisfied, the central limit theorem for normalized sums was proved in [4].

Proof. First of all, it is quite easily verified that

$$\text{L1, L6} \to \text{L3}; \qquad \text{L1, L4, L7} \to \text{L3}; \qquad \text{L8} \to \text{L7}; \qquad \text{L9, L4} \to \text{L3}.$$

These relations prove the corollaries. In proving the theorem it is convenient to introduce the random variables $\xi_{nk} = \xi_k / n^{1/2q}$ with characteristic functions f_{nk} $(1 \leqslant k \leqslant n)$. Let $\varphi_n(t)$ be the characteristic function of the variable

$$\eta_n = \sum_{k=1}^{n} \xi_{nk}.$$

It follows from L3 that $\ln \varphi_n(t)$ exists and

$$\ln \varphi_n(t) = \sum_{k=1}^{n} (f_{nk}(t) - 1) + R_n(t),$$

where

$$|R_n(t)| \leqslant \max_{1 \leqslant k \leqslant n} |f_{nk}(t) - 1| \cdot \sum_{k=1}^{n} |f_{nk}(t) - 1|.$$

From conditions L1, L2 it follows that

$$\sum_{k=1}^{n} |f_{nk}(t) - 1| \leqslant C \frac{t^{2q}}{(2q)!}.$$

Thus, if conditions L1, L2, L3 are satisfied, then $|R_n(t)| \to 0$ uniformly in t on every finite interval. From condition L1,

$$\sum_{k=1}^{n}(f_{nk}(t)-1) = -\frac{1}{n}\frac{t^{2q}}{(2q)!}\sum_{k=1}^{n}(a_k+ib_k)+\rho_n(t),$$

where

$$\rho_n(t) = \sum_{k=1}^{n}\int\left(e^{itx}-1-\dots-\frac{(itx)^{2q}}{(2q)!}\right)dF_{nk}(x).$$

From conditions L2, L4, $|\rho_n(t)| \to 0$ uniformly on every finite interval. Using also condition L5, we obtain that

$$\varphi_n(t) \to \exp\left[-\frac{t^{2q}}{(2q)!}(a+ib)\right]$$

uniformly on every finite interval, and therefore also in the weak sense over K. The theorem is proved.

A sequence of random variables $\xi_1, \dots, \xi_k, \dots$ is called **convergent to 0** if the sequence of the corresponding characteristic functions $f_n(t) \Rightarrow 1$, or, equivalently, if $F_n(x) \Rightarrow \varepsilon(x)$.

Law of large numbers. Chebyshev' s theorem. Let $\xi_1, \dots, \xi_k, \dots$ be a sequence of independent random variables, and let $F_1, \dots, F_k, \dots$ be the sequence of the corresponding distribution functions. If the following conditions are satisfied:

T1. For $k = 1, 2, \dots$ and some integer $p \geqslant 1$,

$$\int x\,dF_k(x) = \dots = \int x^{p-1}\,dF_k(x) = 0; \qquad \int x^p\,dF_k(x) = c_k + id_k.$$

T2. For some $\alpha > 0$, as $n \to \infty$,

$$\frac{1}{n^{\alpha p}}\sum_{k=1}^{n}\int |x|^p\,dV(F_k, x) \to 0,$$

then the sequence of random variables

$$\eta_n = \frac{1}{n^\alpha}\sum_{k=1}^{n}\xi_k$$

tends to 0 as $n \to \infty$.

Proof. Let $\varphi_n(t)$ be the characteristic function of η_n. Introduce the quantities $\xi_{nk} = \xi_k/n^\alpha$. From conditions T1, T2,

$$\max_{1\le k\le n} |f_{nk}(t) - 1| \le \frac{t^p}{p!} \max_{1\le k\le n} \int |x|^p \, dV(F_{nk}, x) \to 0 \qquad (n\to\infty).$$

Consequently, there exists and is finite

$$\ln \varphi_n(t) = \ln \prod_{k=1}^{n} f_{nk}(t).$$

When

$$\max_{1\le k\le n} |f_{nk}(t) - 1| < \frac{1}{2},$$

the inequality

$$|\ln \varphi_n(t)| \le \left(1 + \max_{1\le k\le n} |f_{nk}(t) - 1|\right) \sum_{k=1}^{n} |f_{nk}(t) - 1|$$

is valid. From conditions T1, T2,

$$\sum_{k=1}^{n} |f_{nk}(t) - 1| \le \frac{t^p}{p!} \sum_{k=1}^{n} \int |x|^p \, dV(F_{nk}, x) \to 0.$$

Thus, $\varphi_n(t) \to 1$ uniformly in t on every finite interval. The theorem is proved.

Remark. For $p = 2$, $\alpha = 1$, and real nondecreasing $F_1, \dots, \dots, F_k, \dots$, we obtain the classical Chebyshev theorem.

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REFERENCES

1. I. M. Gelfand, G. E. Shilov, *Generalized Functions and Operations on Them*, Moscow, 1958.

2. I. M. Gelfand, G. E. Shilov, *Some Questions in the Theory of Differential Equations*, Moscow, 1958.

3. Yu. V. Linnik, *Decompositions of Probability Laws*, Leningrad, 1960.

4. V. Yu. Krylov, DAN, **139**, No. 1 (1961).

Note: Figure translations are in progress. See original paper for figures.