CLASSES OF WELL-POSED SOLVABILITY OF A BOUNDARY-VALUE PROBLEM IN AN INFINITE LAYER

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Abstract

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CLASSES OF WELL-POSED SOLVABILITY OF A BOUNDARY-VALUE PROBLEM IN AN INFINITE LAYER

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We shall consider equations of the form

$$\partial^2 u(x,t)/\partial t^2 + P(\partial/\partial x)\partial u(x,t)/\partial t + Q(\partial/\partial x)u(x,t) = 0, \tag{1}$$

$$0 < t < T, \qquad x = (x_1, \dots, x_n), \qquad -\infty < x_i < \infty, \qquad i = 1, \dots, n,$$

with boundary conditions

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x), \qquad -\infty < x_i < \infty, \qquad i = 1, \dots, n. \tag{2}$$

Here $P(\partial/\partial x), Q(\partial/\partial x)$ are polynomials in $\partial/\partial x_1, \dots, \partial/\partial x_n$ with constant coefficients; $u_0(x), u_T(x)$ are prescribed functions. By a solution of problem (1)–(2) we shall mean (continuous) functions $u(x,t)$ satisfying equation (1) and conditions (2) in the sense of generalized functions (^{1}, pp. 129–130). We investigate the question: under what conditions on the smoothness, as well as on the growth as $|x| \to \infty$, of the functions $u_0(x)$ and $u_T(x)$ and their derivatives does a solution of problem (1)–(2) (in the above sense) exist and is unique* in one or another class of functions, and also depend continuously on the functions $u_0(x)$ and $u_T(x)$ in a sense that will be specified each time.

1°. **Definition 1.** Problem (1)–(2) is called **well-posedly solvable in the class of bounded functions** if:
a) for any $l \geq 0$ there exists an $l_1 \geq 0$ such that, for any functions $u_0(x)$ and $u_T(x)$ bounded for all $x \in R_n$ together with all their derivatives up to order l_1, there exists a unique solution $u(x,t)$ of problem (1)–(2), which for each $t \in (0,T)$ is bounded together with all its derivatives up to order l;
b) if the functions $u_{0m}(x)$ and $u_{Tm}(x)$ and their derivatives up to order l_1 tend (uniformly in R_n) to zero as $m \to \infty$, then the corresponding solutions $u_m(x,t)$

of problem (1)–(2), for each $t \in (0, T)$, also tend uniformly to zero together with their derivatives up to order l.

Introduce the following notation. Let Z be the set of those points $s = \sigma + i\tau = (\sigma_1 + i\tau_1, \dots, \sigma_n + i\tau_n)$ for which

$$D(s) \equiv \frac{1}{4} P^2(-is) - Q(-is) = -k^2\pi^2/T^2$$

for some integer $k \geq 1$. Denote

$$\Lambda(s) = |\operatorname{Re} P(is)| - 2|\operatorname{Re} \sqrt{D(-s)}|.$$

Definition 2. Equation (1) will be called **correct** if:
a) real values $s = \sigma$ do not belong to the set Z;
b) the function $\Lambda(s)$ is bounded above for all real $s = \sigma$: $\Lambda(\sigma) \leq C$, $-\infty < \sigma_i < \infty$, $i = 1, \dots, n$. For example, any equation (1) for which $P(s) \equiv 0$ (or $P(s) \equiv \text{const}$) will be correct if condition a) is fulfilled for it. Thus the equations

$$\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 + \partial u/\partial x = 0, \qquad \partial^2 u/\partial t^2 + \partial^2 u/\partial x^2 = 0,$$

$$\partial^2 u/\partial t^2 \pm \partial u/\partial x = 0 \tag{1'}$$

are correct, while the equation $\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 = 0$ is not correct.

The naturalness of singling out the class of correct equations is shown by the following

Theorem 1. *In order that problem (1)–(2) be correctly solvable in the class of bounded functions, it is necessary and sufficient that equation (1) be correct.*

2°. One can also indicate broader classes of functions (in the sense of growth at infinity) in which problem (1)–(2) for a correct equation (1) is correctly solvable.

Lemma 1. *If equation (1) is correct, then there exist constants $A > 0$ and α such that in the domain $\Delta_{\alpha,A}$*

$$\Delta_{\alpha,A} = \{s = \sigma + i\tau : \ |\tau| \leq A(1 + |\sigma|)^\alpha, \ -\infty < \sigma_i < \infty, \ i = 1, \dots, n\} \tag{3}$$

two conditions are satisfied: 1) $\Delta_{\alpha,A} \cap Z = \emptyset$; 2) there exists C_1 such that $\Lambda(s) \leq C_1$ for $s \in \Delta_{\alpha,A}$.

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[*] The question of uniqueness of the solution of problem (1)–(2) is the subject of the author' s note (²).

Definition 3. We shall call a correct equation (1) an **equation of zero type** if the result of Lemma 3 holds with $\alpha \geq 0$, and an **equation of negative type** otherwise. We shall say that a correct equation of negative type has type α_0 (< 0), if α_0 is the exact upper bound of those values of α for which the result of Lemma 1 is valid in $\Delta_{\alpha,A}$.

Denote by $M_{\beta,B}$ the class of (locally integrable) functions $f(x)$ satisfying the estimate

$$|f(x)| \leq C_f \exp\{B|x|^\beta\}. \tag{4}$$

Theorem 2. *Let (1) be a correct equation of zero type, and let $A_0 > 0$ be the exact upper bound of those A for which, in the domain $\Delta_{0,A}$, the result of Lemma 1 is valid. Then for any $l \geq 0$ one can indicate an $l_1 \geq 0$ such that, for any functions $u_0(x)$, $u_T(x)$ which belong to the class $M_{1,B}$ $(B < A_0)$ together with all their derivatives up to order l_1, there exists in the class $M_{1,B}$ a unique solution $u(x,t)$ of problem (1)–(2); this solution has derivatives up to order l (with respect to x) belonging to $M_{1,B}$ and depends continuously on the functions $u_0(x)$, $u_T(x)$ in the following sense: if in the estimate*

$$\max_{k \leq l_1} \max\left\{\left|\partial^k u_{0m}(x)/\partial x_1^{k_1} \cdots \partial x_n^{k_n}\right|,\ \left|\partial^k u_{Tm}(x)/\partial x_1^{k_1} \cdots \partial x_n^{k_n}\right|\right\} \leq C_m \exp\{B|x|\},$$

$$m = 1, 2, \dots,$$

$C_m \to 0$ as $m \to \infty$, then in the corresponding estimate of the solution and its derivatives

$$\max_{k \leq l} |D^{(k)} u_m(x,t)| \leq C_{mt} \exp\{B|x|\}, \quad t \in (0,T),$$

also $C_{mt} \to 0$ as $m \to \infty$.

For example, the first of equations (1′) is a correct equation of zero type. For it $A_0 = 1/2$, if $T \geq 2\pi$, and $A_0 = 1/2 - \sqrt{1/4 - \pi^2/T^2}$, if $0 < T \leq 2\pi$.

Theorem 3. *Let (1) be a correct equation of negative type. Then for any $l \geq 0$ and $m \geq 0$ there exists an $l_1 \geq 0$ such that if the functions $u_0(x)$, $u_T(x)$ have polynomial order of growth m together with all their derivatives up to order l_1:*

$$\max_{k \leq l} \max\{|D^{(k)} u_0(x)|,\ |D^{(k)} u_T(x)|\} \leq C(1+|x|)^m, \tag{5}$$

then problem (1)–(2) has a (unique) solution $u(x,t)$, which together with its derivatives up to order l has polynomial order of growth not exceeding $m+n+1$:

$$\max_{k\le l}|D^{(k)}u(x,t)| \le C_t(1+|x|)^{m+n+1}, \quad 0<t<T. \tag{6}$$

This solution depends continuously on the initial functions $u_0(x)$, $u_T(x)$ in the sense that C_t in (6) depends on C in (5) and tends to zero as $C \to 0$.

3°. **Definition 4.** We shall call the correct equation (1) **strongly correct** if there exist constants $C_1 > 0$, $h > 0$, and C_2 such that

$$\Lambda(\sigma) \le -C_1|\sigma|^h + C_2, \qquad -\infty < \sigma_i < \infty, \quad i = 1,\dots,n. \tag{7}$$

For example, the last two of equations ($1'$) are strongly correct; for the first of them $h = 1$, for the second $h = \frac{1}{2}$.

For strongly correct equations there is a result analogous to that formulated in Lemma 1.

Lemma 2. There exists a domain $\Delta_{\alpha,A}$ of the form (3) such that $\Delta_{\alpha,A} \cap Z = \emptyset$, and in $\Delta_{\alpha,A}$ the estimate (7) is preserved, possibly with C_1 and C_2 replaced by some other constants $\hat{C}_1 > 0$ and $\hat{C}_2 > 0$, and h by any $h_1 < h$.

This fact makes it possible, analogously to Definition 3, to introduce a classification of strongly correct equations, distinguishing equations of zero kind and equations of negative kind α_0.

Definition 5. We shall say that the problem (1)–(2) is **correctly solvable in the class of functions** $M_{\beta,B}$ if: 1) for any functions $u_0(x) \in M_{\beta,B}$ and $u_T(x) \in M_{\beta,B}$ there exists a unique solution $u(x,t)$ of problem (1)–(2), which for each $t \in (0,T)$ is a function of the class $M_{\beta,B}$; 2) if

$$|u_{0m}(x)| \le C_m \exp\{B|x|^\beta\}, \qquad |u_{Tm}(x)| \le C_m \exp\{B|x|^\beta\}$$

and $C_m \to 0$ as $m \to \infty$, then the corresponding solutions $u_m(x,t)$ of problem (1)–(2) satisfy the estimates

$$|u_m(x,t)| \le C_{tm} \exp\{B|x|^\beta\},$$

where $C_{tm} \to 0$ as $m \to \infty$ $(0<t<T)$.

Theorem 4. Let (1) be a strongly correct equation of zero kind, and let $A_0 > 0$ be the exact upper bound of those A for which, in the domain $\Delta_{0,A}$, the result of Lemma 2 is valid. Then the problem (1)–(2) is correctly solvable in any class $M_{1,B}$, where $B < A_0$, and also in any class $M_{\beta,B}$, where $0 < \beta < 1$, $B > 0$ is arbitrary.

For example, the second and third equations in (1) belong to the class of strongly correct equations of zero kind. For the first of them $A_0 = \pi/T$, for the second $A_0 = \pi^2/T^2$.

Theorem 5. Let equation (1) be a strongly correct equation of negative kind α. Then the problem (1)–(2) is correctly solvable in the class of functions $M_{\beta,B}$ for $\beta < h/(h-\alpha)$ and any $B > 0$.

4°. Theorems 4 and 5 show that correct solvability of the problem (1)–(2) for strongly correct equations takes place in classes of increasing functions without assumptions concerning the smoothness of the functions $u_0(x)$ and $u_T(x)$. It can be shown that in this case the solution of the problem (1)–(2) turns out to be an (infinitely) differentiable (with respect to x) function for $t \in (0, T)$. This property proves to be characteristic for strongly correct equations. Namely, the following holds.

Theorem 6. If, with respect to an equation of the form (1) (for $n = 1$), it is known that: a) the problem (1)–(2) for it is correctly solvable in some class of functions M, containing all bounded functions of the first Baire class, i.e., for any $u_0(x) \in M$ and $u_T(x) \in M$ there exists one and only one solution $u(x,t) \in M$, and from the fact that the sequences

$$u_{0m}(x) \longrightarrow 0, \qquad u_{Tm}(x) \longrightarrow 0 \quad (m \to \infty)$$

uniformly on any interval and bounded on the whole line, it follows that the corresponding sequence of solutions

$$u_m(x,t) \longrightarrow 0$$

in the same sense; b) the solution $u(x,t)$ of the problem (1)–(2) for any $t > 0$ is a continuously differentiable function, then equation (1) is strongly correct.

5°. We now single out those correct equations for which correct solvability of the problem (1)–(2) takes place in the class of all sufficiently smooth functions without restriction on their growth as $|x| \to \infty$.

Theorem 7. In order that problem (1)–(2) have a unique solution $u(x,t)$ for arbitrary functions $u_0(x)$ and $u_T(x)$ possessing derivatives up to some order l, and that from the fact that the sequences of functions $u_{0m}(x)$, $u_{Tm}(x)$ tend to zero as $m \to \infty$, together with all their derivatives up to order l, uniformly in every bounded domain, it should follow that the corresponding sequence of solutions $u_m(x,t)$, for each $t \in (0,T)$, also tends uniformly to zero in every bounded domain as $m \to \infty$, it is necessary and sufficient that equation (1) have the form

$$\left(\frac{\partial}{\partial t} + \sum_{k=1}^{n} A_k \frac{\partial}{\partial x_k} + B\right)\left(\frac{\partial}{\partial t} + \sum_{k=1}^{n} A_k \frac{\partial}{\partial x_k} + C\right) u(x,t) = 0,$$

where $A_1, \dots, A_n$ are arbitrary real constants, and B and C are complex constants, $T(B - C) \neq 2k\pi i,\ k = \pm 1, \pm 2, \dots$.

Let us finally note that if equation (1) is not well-posed, then the existence of a solution of problem (1)–(2) can be guaranteed under the assumption that the functions $u_0(x)$ and $u_T(x)$ are continued into the complex space $z = x + iy$ in such a way that entire functions of exponential type are obtained, depending on equation (1).

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2. V. M. Borok, DAN, 183, No. 5 (1968).

Note: Figure translations are in progress. See original paper for figures.

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