SOME CASES IN WHICH FOURIER SUMS GIVE AN APPROXIMATION OF THE ORDER OF THE BEST

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Abstract

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G. I. NATANSON

SOME CASES IN WHICH FOURIER SUMS GIVE AN APPROXIMATION OF THE ORDER OF THE BEST

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It is known that for functions of the class W_ω^{rH} (i.e., 2π-periodic functions having r-th derivatives, whose modulus of continuity $\omega(f^{(r)}, \delta)$ does not exceed the true majorant of moduli of continuity * $\omega(\delta)$), the deviation of the partial sums of the trigonometric Fourier series has order $n^{-r}\omega(n^{-1})\ln n$, and for the whole class W_ω^{rH} this order is definitive.

In the present note it is shown that for certain rather broad subclasses of the class W_ω^{rH} the factor $\ln n$ can be omitted. In other words, for functions from such subclasses the order of the deviation of Fourier sums coincides with the order of the quantity $\sup\limits_{f\in W_\omega^{rH}} E_n(f)$, where $E_n(f)$ is the best approximation of the function f by polynomials

$$A_0 + \sum_{k=1}^{n}(A_k \cos kx + B_k \sin kx).$$

Lemma 1 (see [1,2]). *If $f \in W_\omega^{rH}$ and $S_n[f;x]$ is the n-th partial sum of the trigonometric Fourier series of f, then*

$$S_n[f;x]-f(x) = \frac{n^{1-r}}{2\pi^2}\sum_{k=1}^{m}\frac{1}{k}\int_{y_k}^{y_{k+1}}\left[f^{(r)}\left(x+\frac{r'\pi}{2n}+t\right)+f^{(r)}\left(x+\frac{r'\pi}{2n}-t\right)\right]\times$$

$$\times \sin nt\,dt + (r+1)O(n^{-r}\omega(n^{-1})),$$

where $m = (n-3)/2$, $y_k = (4k+1)\pi/2n$, $r' \in (-2, 2]$, $r' \equiv r \pmod 4$; the constant entering the O-term is absolute.

Lemma 2 [3]. *If the majorant ω satisfies the condition*

$$\delta \int_\delta^\pi \frac{\omega(t)}{t^2}\,dt = O(\omega(\delta)), \tag{*}$$

then there exist constants A and $\mu \in (0,1)$ such that $\delta_1^\mu \omega(\delta_2) \le A\delta_2^\mu \omega(\delta_1)$ for $0 < \delta_1 < \delta_2 \le \pi$.

Theorem 1. *Let $f \in H_\omega = W^0 H_\omega$ and be monotone on the intervals $[c_i, c_{i+1}]$, where $c_0 < c_1 < \dots < c_N = c_0 + 2\pi$. If ω satisfies condition $(*)$, then*

$$S_n[f;x] - f(x) = O(\omega(n^{-1})\ln(N+1)),$$

where the constant entering the O-term depends only on ω.

Proof. On the basis of Lemma 1 we have

$$S_n[f;x] - f(x) = \frac{n}{2\pi^2}\left\{\sum_{k=1}^{m}\frac{1}{k}\int_{y_k}^{y_{k+1}} f(x+t)\sin nt\,dt + \right.$$

$$\left. + \sum_{k=1}^{m}\frac{1}{k}\int_{y_k}^{y_{k+1}} f(x-t)\sin nt\,dt\right\} + O\left(\omega\left(\frac{1}{n}\right)\right).$$

* $\omega(\delta)$ is a true majorant of moduli of continuity if: 1) ω is defined on $[0,+\infty)$; 2) $\lim\limits_{\delta\to 0}\omega(\delta) = \omega(0) = 0$; 3) for $0 \le \delta_1 < \delta_2$ one has $0 \le \omega(\delta_2) - \omega(\delta_1) \le \omega(\delta_2 - \delta_1)$.

Let $k_0, k_1, \dots, k_{p+1}$ be the numbers of those intervals $[y_k, y_{k+1}]$ into which at least one of the points $y_1, y_m, c_i - x$ falls (or a point differing from $c_i - x$ by an integral multiple of 2π). Obviously, $p \le N$. Then (as usual, $\sum_{k=\mu}^{\nu} a_k = 0$ if $\mu > \nu$)

$$\Sigma_1 = \sum_{k=1}^{m}\frac{1}{k}\int_{y_k}^{y_{k+1}} f(x+t)\sin nt\,dt = \sum_{i=0}^{p}\sum_{k=k_i+1}^{k_{i+1}-1}\frac{1}{k}\int_{y_k}^{y_{k+1}} f(x+t)\sin nt\,dt +$$

$$+ \sum_{i=0}^{p+1}\frac{1}{k_i}\int_{y_{k_i}}^{y_{k_i+1}} f(x+t)\sin nt\,dt = \Sigma_1' + \Sigma_1''.$$

Since

$$\int_{y_k}^{y_{k+1}} f(x+t)\sin nt\,dt = O(n^{-1}\omega(n^{-1})),$$

we have

$$\Sigma_1'' = O\left(\frac{1}{n}\,\omega\left(\frac{1}{n}\right)\right)\sum_{i=0}^{p+1}\frac{1}{k_i} = O\left(\frac{1}{n}\,\omega\left(\frac{1}{n}\right)\right)\sum_{i=1}^{N+2}\frac{1}{i} = O\left(\frac{\ln(N+1)}{n}\,\omega\left(\frac{1}{n}\right)\right).$$

Since $f(x+\cdot)$ is monotone on $[y_{k_i+1}, y_{k_{i+1}}]$, for $k_i < k < k_{i+1}$ we have

$$\left|\int_{y_k}^{y_{k+1}} f(x+t)\sin nt\,dt\right| \leq \frac{2\pi}{n}\left|\Delta_{2\pi/n} f(x+y_k)\right|,$$

whence

$$|\Sigma_1'| \leq \frac{2\pi}{n}\sum_{i=0}^{p}\left|\sum_{k=k_i+1}^{k_{i+1}-1}\frac{1}{k}\Delta_{2\pi/n} f(x+y_k)\right|.$$

But

$$\sum_{k=k_i+1}^{k_{i+1}-1}\frac{1}{k}\Delta_{2\pi/n} f(x+y_k) = \sum_{k=k_i+1}^{k_{i+1}-1}\frac{f(x+y_k)-f(x+y_{k_i+1})}{(k-1)k} +$$

$$+\frac{f(x+y_{k_i+1})-f(x+y_{k_{i+1}})}{k_{i+1}-1},$$

$$|f(x+y_k)-f(x+y_{k_i+1})| \leq \omega\left(\frac{2(k-1)}{n}\pi\right),$$

$$|f(x+y_{k_i+1})-f(x+y_{k_{i+1}})| \leq \omega\left(\frac{2(k_{i+1}-1-k_i)}{n}\pi\right).$$

Thus,

$$|\Sigma_1'| \leq \frac{4\pi}{n}\sum_{i=1}^{p}\left[\sum_{k=k_i+1}^{k_{i+1}-1}\frac{\omega((k-1)\pi n^{-1})}{(k-1)k} + \frac{\omega((k_{i+1}-1-k_i)\pi n^{-1})}{k_{i+1}-1}\right] \leq$$

$$\leq \frac{8\pi}{n}\sum_{k=2}^{m}\frac{\omega((k-1)\pi n^{-1})}{k^2} + \frac{4\pi}{n}\sum_{i=0}^{p}\frac{\omega((k_{i+1}-1-k_i)\pi n^{-1})}{k_{i+1}-1}.$$

The first of the resulting sums, by virtue of $(*)$, is $O(n^{-1}\omega(n^{-1}))$.
To estimate the second sum we apply Lemma 2:

$$\frac{4\pi}{n}\sum_{i=0}^{p}\frac{\omega((k_{i+1}-1-k_i)\pi n^{-1})}{k_{i+1}-1}=O\Big(\frac{1}{n}\,\omega\Big(\frac{1}{n}\Big)\Big)\sum_{i=0}^{p}\frac{(k_{i+1}-1-k_i)^{\mu}}{k_{i+1}-1}.$$

Denote by l such a number that $k_l \le (p+1)^{1/(1-\mu)}+1 < k_{l+1}$ (if all the numbers $k_1, k_2, \dots, k_{p+1}$ lie on one side of $(p+1)^{1/(1-\mu)}+1$, then

below one of the sums vanishes). Then

$$\sum_{i=0}^{p}\frac{(k_{i+1}-1-k_i)^{\mu}}{k_{i+1}-1}\ll\sum_{i=0}^{l-1}\frac{k_{i+1}-1-k_i}{k_{i+1}-1}+\sum_{i=l}^{p}(k_{i+1}-1)^{\mu-1}<$$

$$<\sum_{i=0}^{l-1}\sum_{k=k_i}^{k_{i+1}-1}\frac{1}{k}+\sum_{i=0}^{p}[(p+1)^{1/(1-\mu)}]^{\mu-1}\ll\frac{1}{1-\mu}\ln(p+1)+2.$$

Thus,

$$\Sigma_1'=O\left(\frac{1}{n}\omega\left(\frac{1}{n}\right)\right)\left(1+\frac{1}{1-\mu}\ln(p+1)+2\right)=O\left(\frac{\ln(N+1)}{n}\omega\left(\frac{1}{n}\right)\right).$$

Consequently,

$$\Sigma_1=O\left(\ln(N+1)\omega n^{-1}/n\right).$$

Similarly,

$$\Sigma_2=\sum_{k=1}^{m}\frac{1}{k}\int_{y_k}^{y_{k+1}}f(x-t)\sin nt\,dt=O\left(\frac{\ln(N+1)}{n}\omega\left(\frac{1}{n}\right)\right).$$

The theorem is proved.

Remark 1. If $\omega(\delta)=\delta^{\mu}$, where $0<\mu<1$, then

$$S_n[f;x]-f(x)=O\left(\frac{\ln(N+1)}{(1-\mu)n^{\mu}}\right),$$

and the constant entering the O-term is absolute.

Remark 2. The condition of piecewise monotonicity of f cannot be replaced by the condition of bounded variation.

Remark 3. The condition $(*)$ imposed on the majorant ω is essential.

Lemma 3. *If f is convex upward on $[y_k, y_{k+1}]$, then*

$$\frac{4}{\pi n}\Delta^2_{\pi/n} f(y_k) \ll \int_{y_k}^{y_{k+1}} f(t) \sin nt \, dt \ll 0.$$

Proof. We shall show that

$$-\frac{8}{\pi} g(\pi) \ll \int_0^{2\pi} g(z) \cos z \, dz \ll 0,$$

if g is convex upward on $[0, 2\pi]$, $g(0) = g(2\pi) = 0$. Let

$$h(z) = [g(z) + g(2\pi - z)]/2, \qquad l(z) = 2h(\pi/2)z/\pi.$$

Then

$$\int_0^{2\pi} g(z) \cos z \, dz = 2 \int_0^{\pi} h(z) \cos z \, dz \gg$$

$$\gg 2 \int_0^{\pi} l(z) \cos z \, dz = -\frac{8}{\pi} h\left(\frac{\pi}{2}\right) \gg -\frac{8}{\pi} g(\pi).$$

The inequality

$$\int_0^{2\pi} g(z) \cos z \, dz \ll 0$$

is obvious. To complete the proof it is enough to put

$$f\left(\frac{z}{n} + y_k\right) - \frac{z}{2\pi}\left[f(y_{k+1}) - f(y_k)\right] - f(y_k) = g(z)$$

and to note that

$$g(\pi) = -\frac{1}{2}\Delta^2_{\pi/n} f(y_k).$$

Theorem 2. *If $f \in H_\omega$ and is convex (upward or downward) on each of the intervals $[c_i, c_{i+1}]$, where $c_0 < c_1 < \cdots < c_N = c_0 + 2\pi$, then*

$$S_n[f; x] - f(x) = O(\omega(n^{-1}) \ln(N+1)),$$

where the constant entering the O-term is absolute.

Proof. With the preceding notation it is enough to show that

$$\Sigma_1' = O\left(\ln(N+1)\omega n^{-1}/n\right).$$

The function $f(x+\cdot)$ preserves the direction of convexity on

$$[y_{k_i+1}, y_{k_{i+1}}].$$

Hence, by Lemma 3,

$$|\Sigma_1'| \ll \sum_{i=0}^{p}\left|\sum_{k=k_i+1}^{k_{i+1}-1}\frac{1}{k}\int_{y_k}^{y_{k+1}} f(x+t)\sin nt\,dt\right| \ll \frac{4}{\pi n}\sum_{i=0}^{p}\left|\sum_{k=k_i+1}^{k_{i+1}-1}\frac{1}{k}\Delta^2_{\pi/n} f(y_k)\right|.$$

Moreover, $y_{k+1/2} = (4k+3)\pi/2n$,

$$|\Sigma_1'| \le \frac{8}{\pi n}\sum_{i=0}^{p}\left|\sum_{\nu=2k_i+2}^{2k_{i+1}-2}\frac{1}{\nu}\Delta^2_{\pi/n} f(y_{\nu/2})\right|.$$

Next

$$\sum_{\nu=2k_i+2}^{2k_{i+1}-2}\frac{1}{\nu}\Delta^2_{\pi/n} f(y_{\nu/2}) = \sum_{\nu=2k_i+2}^{2k_{i+1}-2}\frac{\Delta_{\pi/n} f(y_{\nu/2})}{(\nu-1)\nu} - \frac{\Delta_{\pi/n} f(y_{k_i+1})}{2k_i+1} + \frac{\Delta_{\pi/n} f(y_{k_{i+1}-1/2})}{2k_{i+1}-2}$$

and $|\Delta_{\pi/n} f(x)| \le \omega(\pi/n)$. Therefore

$$\left|\sum_{\nu=2k_i+2}^{2k_{i+1}-2}\frac{1}{\nu}\Delta^2_{\pi/n} f(y_{\nu/2})\right| \le \omega\left(\frac{\pi}{n}\right)\left[\sum_{\nu=2k_i+1}^{2k_{i+1}}\frac{1}{(\nu-1)\nu} + \frac{1}{k_i}\right].$$

Thus,

$$|\Sigma_1'| \le \frac{8}{\pi n}\omega\left(\frac{\pi}{n}\right)\left[\sum_{i=0}^{p}\sum_{\nu=2k_i+1}^{2k_{i+1}}\frac{1}{(\nu-1)\nu} + \sum_{i=0}^{p}\frac{1}{k_i}\right] = O\left(\frac{\ln(N+1)}{n}\omega\left(\frac{1}{n}\right)\right),$$

which was required to be proved.

We shall call a function f q-monotone on $[a,b]$ if, for all x and $h>0$ such that $x, x+qh \in [a,b]$,

$$\Delta_h^q f(x) = \sum_{\nu=0}^{q} (-1)^{q+\nu} C_q^\nu f(x+\nu h) \geq 0 \ (\leq 0).$$

It is not difficult to verify that: a) if f is q-monotone on $[a,b]$ and f' exists, then f' is $(q-1)$-monotone on $[a,b]$; b) if a continuous f is q-monotone on $[a,b]$, then either f is $(q-1)$-monotone on $[a,b]$, or there exists a $c \in (a,b)$ such that f is $(q-1)$-monotone on $[a,c]$ and $[c,b]$.

Theorem 3. Let $f \in WrH_\omega$ and let f be q-monotone on the intervals $[c_i, c_{i+1}]$, where $c_0 < c_1 < \cdots < c_N = c_0 + 2\pi$. Suppose, further, that $r \leq q-1$, and for $r = q-1$ condition $(*)$ is satisfied for ω. Then

$$S_n[f;x] - f(x) = O((q + \ln(N+1))\omega(n^{-1})n^{-r}),$$

where the constant entering the O-term is absolute for $r < q-1$ and depends only on ω for $r = q-1$.

Proof. For $r = q-1$ the function $f^{(r)}$ is monotone on the intervals $[c_i, c_{i+1}]$. The proof of Theorem 1 (with insignificant modifications) gives the required result. If $r < q-1$, then $f^{(r)}$, being $(q-r)$-monotone on $[c_i, c_{i+1}]$, is convex on the intervals $[d_i, d_{i+1}]$, where $d_0 < d_1 < \cdots < d_M = d_0 + 2\pi$ and $M \leq 2^{q-r-2}N$. Slightly modifying the proof of Theorem 2, here also we obtain the needed estimate.

Assertions analogous to those proved are apparently also valid for Fourier series in Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. We confine ourselves to formulating the following result obtained by us.

Let $0 < \mu \leq 1$, $-1/2 < \alpha, \beta < 1/2$, and let $S_n^{(\alpha,\beta)}[f;x]$ be the n-th partial sum of the Fourier-Jacobi series of a function f defined on $[-1,1]$. Then, as shown in ([4]),

$$\sup_{f \in \mathrm{Lip}\,1\mu} \left| S_n^{(\alpha,\beta)}[f;1] - f(1) \right| \asymp n^{\alpha+1/2-\mu}.$$

If, however, $f \in \mathrm{Lip}_1 \mu$ and the function $f(\cos\cdot)$ is convex on the intervals $[c_i, c_{i+1}]$, where $0 = c_0 < c_1 < \cdots < c_N = \pi$, then

$$S_n^{(\alpha,\beta)}[f;1] - f(1) = O\left(Nn^{\alpha-1/2-\mu} + N^{\alpha+1/2+\mu}n^{-2\mu} + n^{-\mu}\right).$$

Leningrad State University
named after A. A. Zhdanov

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.