APPROXIMATION BY ANALYTIC FUNCTIONS IN THE MEAN

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Abstract

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APPROXIMATION BY ANALYTIC FUNCTIONS IN THE MEAN

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The aim of this article is to describe measurable subsets E of the complex plane C having the property that every function $f \in L^p(E)$, analytic at the interior points of the set E, admits approximation in $L^p(E)$ by functions analytic in a neighborhood of E (for $p \in (1, 2]$). Interesting results in this direction were recently obtained by S. O. Sinanyan ([1,2]), who, in particular, found necessary and sufficient conditions for the possibility of approximating any function $f \in L^p(E)$ $(p \geq 2)$ by rational fractions under the assumption that the set E is compact. These conditions are formulated in terms of analytic p-capacity, which is an analogue of analytic capacity (Ahlfors capacity). In contrast to ([1,2]), we shall consider approximation on arbitrary Borel sets E (not necessarily compact), and the necessary and sufficient conditions for the possibility of approximation in $L^2(E)$ will be formulated in terms of Cartan' s fine topology.

We shall use the following notation: $L^p(E)$ $(p \in [1, +\infty))$ is the set of all complex functions f, defined on a measurable (Lebesgue) set $E \subset C$, such that

$$\int_E |f|^p < +\infty$$

(the integral with respect to Lebesgue measure is meant); $L^p_a(E)$ is the set of all functions $f \in L^p(E)$ of the form $f = g|_E$, where g is a function analytic in some open set $G \supset E$, and $g|_E$ is its restriction to the set E.

Everywhere below the word capacity means Wiener capacity.

Theorem 1. Let E be a Borel set $(E \subset C)$.
The following assertions are equivalent:

1) for every number $\varepsilon > 0$ and for every function $f \in L^2(E)$ there exists a function $\varphi \in L^2_a(E)$ such that

$$\int_E |f - \varphi|^2 < \varepsilon;$$

2) the closure of the set $C \setminus E$ in Cartan' s fine topology (for the complex plane C) contains almost all points of the set E.

Definitions and properties of Cartan' s fine topology and of other concepts of potential theory occurring below are presented, for example, in the monographs ([3,4]).

Assertion 2) can also be formulated as follows: the set of all points $t \in E$ at which the complement $C \setminus E$ is thin has measure zero.

Theorem 1 can be supplemented in the following way: if assertion 2) is false, then, in order that a function $f \in L^2(E)$ be approximable arbitrarily closely in $L^2(E)$ by functions from $L^2_a(E)$, it is necessary and sufficient that f satisfy the Cauchy-Riemann conditions in the following sense:

$$\int_E f \frac{\partial \lambda}{\partial \bar{z}} = 0$$

for every function λ having generalized derivatives $\partial\lambda/\partial x$ and $\partial\lambda/\partial y$, square summable in C, and equal to zero everywhere in $C \setminus E$.

Theorem 1 follows easily from the following theorem (by $\dot{A}$ is denoted the set of all interior points of the set A, and by ∂A the boundary of the set A).

Theorem 2. Let E be the same as in Theorem 1. The following assertions are equivalent:

1. For every function $f \in L^2(E)$, analytic on $\mathring{E}$, and for every number $\varepsilon > 0$, there exists a function $\varphi \in L^2_a(E)$ such that
$$\int_E |f - \varphi|^2 < \varepsilon.$$
2. The set of all points $t \in \partial E$ not belonging to the closure, in the fine topology, of the set $C \setminus E$, is polar (i.e., has zero outer capacity).

Assertion 2 in Theorem 2 can also be formulated as follows: the set of all points $t \in \partial E \setminus \partial \mathring{E}$ not belonging to the fine closure of the set $C \setminus E$ has measure zero, and the set of all points $t \in \partial \mathring{E}$ not belonging to the fine closure of the set $C \setminus E$ is polar.

It is easy to prove that if G is an open set ($G \subset C$), then $L^2_a(G)$ contains a nonzero function if and only if the set $C \setminus G$ is nonpolar. If the set $C \setminus E$ is polar, then assertion 2 of Theorem 2 means that the set ∂E is also polar (for $C \setminus E$ is then thin everywhere), and $L^2_a(\mathring{E})$, as well as $L^2_a(E)$, contains no nonzero functions.

We outline the proof of Theorem 2. It is easy to see that a function $f \in L^2(G)$, where G is an open set, is orthogonal to $L^2_a(G)$ if and only if there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of infinitely differentiable finite functions in G such that

$$\lim_{n\to\infty}\int_G\left|\frac{\partial\varphi_n}{\partial z}-f\right|^2=0 \qquad \left(\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right);$$

the orthogonality of the function f to the set $L_a^2(G)$ means that

$$\int_G f\bar{g}=0$$

for all $g\in L_a^2(G)$. If $C\setminus G$ has positive capacity, then, as is known ([5]), there exists a function φ having in G generalized partial derivatives $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$, square summable, and such that

$$\lim_{n\to\infty}\int_G\left|\frac{\partial\varphi_n}{\partial z}-\frac{\partial\varphi}{\partial z}\right|^2=0,$$

so that $f(t)=\dfrac{\partial\varphi}{\partial z}(t)$ for almost all $t\in G$. Here the function φ may be regarded as refined in the sense of Deny ([5,6]), i.e., continuous everywhere in G except at the points of a closed set of arbitrarily small capacity. It is now easy to understand that if $f\in L^2(E)$ and is orthogonal to $L_a^2(E)$ (where E is a set satisfying the conditions of Theorem 2), then there exists a function φ, having generalized partial derivatives $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$, square summable in C, refined, equal to zero in $C\setminus E$, and such that

$$\frac{\partial\varphi}{\partial z}(t)=f(t)$$

for almost all $t\in E$ (in the proof of this fact one must take into account that if $\varphi(t)=0$ approximately everywhere in $C\setminus E$, then, by the Borel nature of the set $C\setminus E$, $\varphi(t)=0$ quasieverywhere in $C\setminus E$, and, without loss of generality, it may be assumed that $\varphi(t)=0$ everywhere in $C\setminus E$). But the function φ, being refined quasieverywhere (i.e., everywhere except at the points of a certain polar set), is finely continuous ([5,6]). If assertion 2 of Theorem 2 is true, then $\varphi(t)=0$ for quasievery $t\in(E\setminus\mathring{E})\cup\partial\mathring{E}$. In this case, applying a theorem of Brelot ([5,7]), we conclude that there exists a sequence of infinitely differentiable finite functions $\{\widetilde{\varphi}_n\}_{n=1}^{\infty}$ with supports in $\mathring{E}$ such that

$$\lim_{n\to\infty}\int_E\left|\frac{\partial\widetilde{\varphi}_n}{\partial z}-f\right|^2=0,$$

and therefore f is orthogonal to every function $g\in L^2(E)$ analytic in E, so that assertion 1 is also true. If, on the other hand, assertion 2 is false, then it is easy to construct a function ψ, having square-summable partial derivatives $\partial\psi/\partial x$ and $\partial\psi/\partial y$, refined, equal to zero everywhere in $C\setminus E$, and such that the set

$$\{t:\ t\in(E\setminus\mathring{E})\cup\partial\mathring{E},\ \psi(t)\neq0\}$$

is nonpolar. In such a

in which case $\partial\psi/\partial z$ is a function orthogonal to $L_a^2(E)$, but not orthogonal to all functions $g \in L^2(E)$ analytic in $\overset{\circ}{E}$, and 1) is false, so that our argument is complete.

S. O. Sinanyan observed that the problem of approximation by functions of the class $L_a^p(E)$ in $L^p(E)$ for $p \in [1, 2)$ is essentially different from the analogous problem for $p \in [2, +\infty)$. He showed, in particular [1], that the set $L_a^p(E)$ is dense in $L^p(E)$ if $p \in [1, 2)$, and E is a nowhere dense compact set. S. O. Sinanyan stated the following hypothesis: whatever the compact set E and the number $p \in [1, 2]$ may be, every function $f \in L^p(E)$ analytic on $\overset{\circ}{E}$ admits arbitrarily good approximation (in the metric of the space $L^p(E)$) by functions from $L_a^p(E)$.

In fact, for $p \in (1, 2)$ an even stronger assertion is true.

Theorem 3. *Let E be a measurable set ($E \subset C$), and let the number p satisfy the inequality $1 < p < 2$. Then for any number $\varepsilon > 0$ and any function $f \in L^p(E)$ analytic on $\overset{\circ}{E}$, there exists a function $\varphi \in L_a^p(E)$ such that*

$$\int_E |f - \varphi|^p < \varepsilon.$$

Let us outline the proof of this theorem. Its assertion is obvious if E fills the plane C. Therefore one may assume that $C \setminus E$ is nonempty. Let $g \in L^q(E)$ ($q = p/(p-1)$), and suppose

$$\int_E f\bar{g} = 0$$

for all $f \in L_a^p(E)$. Arguing in the same way as in the proof of Theorem 2, taking into account the known estimate of the gradient of a finite function ψ in terms of $\partial\psi/\partial z$ in L^q ($q \in (1, +\infty)$) (see, for example, [8], pp. 81-89; [9], p. 207) and the nonemptiness of the set $C \setminus E$, we arrive at the following conclusion: there exists a function φ having (in C) generalized partial derivatives $\partial\varphi/\partial x$, $\partial\varphi/\partial y$, summable to the q-th power, and such that $\varphi(t) = 0$ for all $t \in C \setminus E$,

$$\frac{\partial\varphi}{\partial z}(t) = g(t)$$

for almost all $t \in E$. From S. L. Sobolev' s embedding theorem it follows that the function φ may be taken to be continuous. Therefore $\varphi(t) = 0$ for all $t \in (E \setminus \overset{\circ}{E}) \cup \partial E$. Now, relying on the known properties of functions of the class W_q^1 (see, for example, [5]; [10], § 3, Ch. III), it is not difficult to construct a sequence $\{\widetilde{\varphi}_n\}_{n=1}^{\infty}$ of finite infinitely differentiable functions with supports in $\overset{\circ}{E}$ such that

$$\lim_{n\to\infty} \int_E \left| \frac{\partial\varphi}{\partial z} - \frac{\partial\widetilde{\varphi}_n}{\partial z} \right|^q = 0,$$

so that

$$\int_E f\bar{g} = 0$$

for all $f \in L^p(E)$ analytic in $\mathring{E}$. The proof is complete.

We do not know whether the assertion of Theorem 3 is true for $p = 1$. As is seen from the theorem of S. O. Sinanyan quoted above, it is true if E is a compact nowhere dense set. It is easy to prove that it is true even for any bounded measurable set E with empty interior.

The considerations by means of which Theorems 2 and 3 are proved can apparently also be used to solve problems on approximation in the mean by harmonic functions. Thus, for example, the following serves as an analogue of Theorem 3.

Theorem 4. *Let E be a bounded measurable (with respect to m-dimensional Lebesgue measure) subset of Euclidean space R^m $(m \geq 2)$, and let $\mathring{E} = \Lambda$. If $p \in (1, m/(m-2))$, then for any function $f \in L^p(E)$ and for any number $\varepsilon > 0$ there exist an open set $G \supset E$ $(G \subset R^m)$ and a function u harmonic in G, satisfying the inequality*

$$\int_E |f - u|^p < \varepsilon.$$

The problem of approximation by analytic functions in the mean is equivalent to a certain "purely real" problem of approximation by irrotational vector fields. More precisely, the following holds.

Theorem 5. Let E be a measurable set in the complex plane, $p \in (1, +\infty)$.

The following assertions are equivalent:

1) Whatever the (real) vector field $U = (u_1, u_2)$ given on E and satisfying the conditions

$$\int_E |u_1|^p + |u_2|^p < +\infty,$$

$$\operatorname{rot} U \big|_E = 0,$$

there exists a sequence of open sets $\{G_n\}_{n=1}^{\infty}$ containing E and smooth vector fields $\{V_n\}_{n=1}^{\infty}$ such that V_n is defined in G_n,

$$\operatorname{rot} V_n = 0 \quad (n = 1, 2, \ldots), \qquad \lim_{n\to\infty} \|U - V_n\|_E^p = 0.$$

2) Whatever the function $f \in L^p(E)$, analytic on $\mathring{E}$, there exists a sequence of functions $\{f_n\}_{n=1}^{\infty}$ belonging to $L^p_a(E)$ such that

$$\lim_{n\to\infty} \int_E |f - f_n|^p = 0.$$

In the formulation of assertion 1), $\operatorname{rot} U|_E$ denotes the generalized curl of the field U in $\mathring{E}$, and $\| \ \|$ is the norm in R^2. Instead of approximation by irrotational fields in Theorem 5, one could also speak of approximation by solenoidal fields.

It is natural to expect that, for $p \in (2, +\infty)$, theorems analogous to Theorems 1 and 2 are also valid. Progress in this direction, it seems to us, requires the development of a "fine" theory of functions whose generalized gradient is summable to the power $p \in (1, +\infty)$ (in the spirit of the theory of functions of class BL, developed in ([5],[6])).

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Note: Figure translations are in progress. See original paper for figures.