PERTURBATION THEORY FOR SPHEROIDAL OSCILLATIONS OF THE EARTH

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Abstract

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PERTURBATION THEORY FOR SPHEROIDAL OSCILLATIONS OF THE EARTH

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In previous communications (1–3), a perturbation theory was constructed for torsional and radial oscillations of the Earth. This made it possible to examine theoretically the question of the damping of the corresponding oscillations, and also to carry out some numerical experiments connected with the problem of improving realistic models of the Earth. The perturbation theory for spheroidal oscillations is more complicated, since the original system of equations is of sixth order. Below a method will be indicated for obtaining the basic formulas of the perturbation theory for the Earth’ s proper spheroidal oscillations.

In the spherical coordinate system $\mathbf{r}(r, \theta, \varphi)$, the components of the displacement vector $\mathbf{u}(u, v, w)$ and of the additions to the gravitational potential ψ have the form (4)

$$u = U(r) S_n(\theta, \varphi), \qquad v = V(r)\frac{\partial S_n}{\partial \theta}, \qquad w = \frac{V(r)}{\sin\theta}\frac{\partial S_n}{\partial \varphi}, \qquad \psi = P(r) S_n, \tag{1}$$

where S_n is a surface spherical function of order n. Spheroidal oscillations are described by three second-order equations for the displacements (4)

$$\rho\omega^2 u + A = 0, \qquad \rho\omega^2 v + B = 0, \qquad \rho\omega^2 w + C = 0, \tag{2}$$

where ρ is the density,

$$A = \Big\{\rho g \operatorname{div}\mathbf{u} + \rho\frac{\partial\psi}{\partial r} - \rho\left(4\pi G\rho u - \frac{2}{r} g u + g\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial r}\left(\lambda \operatorname{div}\mathbf{u} + 2\mu\frac{\partial u}{\partial r}\right) +$$

$$+\frac{\mu}{r}\frac{\partial e_{r\theta}}{\partial\theta} + \frac{\mu}{r\sin\theta}\frac{\partial e_{r\varphi}}{\partial\varphi} + \frac{\mu}{r}\left(4e_{rr} - 2e_{\theta\theta} - 2e_{\varphi\varphi} + \operatorname{ctg}\theta\, e_{r\theta}\right)\Big\}, \tag{3}$$

$$B=\left\{\frac{\rho}{r}\frac{\partial\psi}{\partial\theta}+\frac{\partial}{\partial r}(\mu e_{r\theta})+\frac{1}{r}\frac{\partial}{\partial\theta}\left(-g\rho u+\lambda\operatorname{div}\mathbf{u}+2\mu e_{\theta\theta}\right)+\frac{\mu}{r\sin\theta}\frac{\partial e_{\theta\varphi}}{\partial\varphi}+\right.$$

$$\left.+\frac{\mu}{r}\left[2\operatorname{ctg}\theta\left(\frac{1}{r}\frac{\partial v}{\partial\theta}-\frac{v}{r}\operatorname{ctg}\theta-\frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi}\right)+3e_{r\theta}\right]\right\}, \tag{4}$$

$$C=\left\{\frac{\rho}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}+\frac{\partial}{\partial r}(\mu e_{r\varphi})+\frac{\mu}{r}\frac{\partial e_{\theta\varphi}}{\partial\theta}+\frac{3\mu}{r}e_{r\varphi}+\right.$$

$$\left.+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(-g\rho u+\lambda\operatorname{div}\mathbf{u}+2\mu e_{\varphi\varphi}\right)+\frac{2\mu}{r}e_{\theta\varphi}\operatorname{ctg}\theta\right\}. \tag{5}$$

Here g is the acceleration of gravity; λ,μ are the Lamé constants ($\lambda=K-\frac{2}{3}\mu$); K is the bulk modulus; G is the gravitational constant; e_{ij} $(i,j=r,\theta,\varphi)$ are the components of the strain tensor in a spherical coordinate system. To system (2) one should add the Poisson equation for ψ

$$\nabla^2\psi-4\pi G\left(\rho\operatorname{div}\mathbf{u}+u\frac{\partial\rho}{\partial r}\right)=0. \tag{6}$$

For concrete calculations it is convenient to pass from system (2)–(6) to a sixth-order system for the radial functions (4)

$$y_1=U,\qquad y_2=\lambda X+2\mu\frac{\partial U}{\partial r},\qquad y_3=V,\qquad X=\frac{\partial U}{\partial r}+\frac{2}{r}U-\frac{n(n+1)}{r}V,$$

$$y_4=\mu\left(\frac{\partial V}{\partial r}-\frac{V}{r}+\frac{U}{r}\right),\qquad y_5=P,\qquad y_6=\frac{\partial P}{\partial r}-4\pi G\rho U. \tag{7}$$

and introduce dimensionless functions z_i $(i=1,2,\dots,6)$. Then, denoting differentiation with respect to the dimensionless radius $x=r/a$, where a is the Earth' s radius, by a dot, we obtain

$$
\begin{aligned}
\dot z_1 =& -(2K_0 - N_0\mu_0)(M_0x)^{-1}z_1 + M_0^{-1}z_2 + n(n+1)(K_0 - N_0\mu_0)(M_0x)^{-1}z_3;\\
\dot z_2 =& [-\chi_0^2\rho_0x^2 - 4\nu\rho_0g_0x + 12N_1\mu_0K_0M_0^{-1}]x^{-2}z_1 - 4N_1\mu_0M_0^{-1}z_2+\\
&+[n(n+1)\nu\rho_0g_0x - 6N_1n(n+1)\mu_0K_0M_0^{-1}]x^{-2}z_3+\\
&+n(n+1)x^{-1}z_4 - \nu\rho_0z_6,\\
\dot z_3 =& -x^{-1}z_1 + x^{-1}z_3 + (N_1\mu_0)^{-1}z_4,\\
\dot z_4 =& [\nu g_0\rho_0x - 6N_1\mu_0K_0M_0^{-1}]x^{-2}z_1 - (K_0 - N_0\mu_0)(M_0x)^{-1}z_2+\\
&+\{-\chi_0^2\rho_0x^2 + 2N_1\mu_0M_0^{-1}[(2n^2+2n-1)(K_0 - N_0\mu_0)+\\
&+2(n^2+n-1)N_1\mu_0]\}x^{-2}z_3 - 3x^{-1}z_4 - \nu\rho_0x^{-1}z_5,\\
\dot z_5 =& D\rho_0z_1 + z_6,\\
\dot z_6 =& -Dn(n+1)\rho_0x^{-1}z_3 + n(n+1)x^{-2}z_5 - 2x^{-1}z_6.
\end{aligned}
\tag{8}
$$

In (8) the following notation has been used:

$$
\begin{gathered}
y_1 = az_1, \quad y_2 = \overline{K}z_2, \quad y_3 = az_3, \quad y_4 = \overline{K}z_4, \quad y_5 = a\overline{g}z_5, \quad y_6 = \overline{g}z_6,\\
K = K_0\overline{K}, \quad \mu = \mu_0\overline{\mu}, \quad \rho = \rho_0\overline{\rho}, \quad g = g_0\overline{g},\\
M_0 = K_0 + N\mu_0, \quad N_1 = \overline{\mu}/\overline{K}, \quad N_0 = \frac{2}{3}N_1,\\
N = \frac{4}{3}N_1, \quad \chi_0^2 = \omega_0^2\overline{\rho}a^2/\overline{K}, \quad \nu = \overline{g}\,\overline{\rho}a/\overline{K}, \quad D = 4\pi G\overline{\rho}a/\overline{g}.
\end{gathered}
\tag{9}
$$

In (9) the barred quantities denote normalizing quantities. In the liquid regions in (8) one should set $\mu_0 = 0$, $z_4 = 0$. The boundary conditions require regularity of all quantities at zero, and at the Earth’ s surface

$$
z_6 + n(n+1)z_5 = 0, \qquad z_2 = z_4 = 0. \tag{10}
$$

The sought functions z_i are continuous everywhere, with the exception of the function z_3, which has a discontinuity at the boundaries of liquid and solid regions. Suppose that problem (8)–(10) has been solved, and let us see how the frequency χ_0 changes,

$$
\chi_0 \to \chi_0 + \chi_1, \qquad \chi_1 \ll \chi_0,
$$

if the properties of the Earth model are slightly changed:

$$
K_0 \to K_0 + K_1, \quad \mu_0 \to \mu_0 + \mu_1, \quad \rho_0 \to \rho_0 + \rho_1, \quad g_0 \to g_0 + g_1 \quad (g_1 = g_1(\rho_1)),
$$

$$(K_1, \mu_1, \rho_1, g_1) \ll (K_0, \mu_0, \rho_0, g_0), \tag{11}$$

with

$$u \to u_0 + u_1, \qquad v \to v_0 + v_1, \qquad w \to w_0 + w_1. \tag{12}$$

Substituting (11) and (12) into (2)–(5) and writing the equations of the first approximation, we represent in them the perturbed functions A, B, C in the form

$$A_1 = A_{11} + A_{12}, \qquad B_1 = B_{11} + B_{12}, \qquad C_1 = C_{11} + C_{12}.$$

The variations of the functions (u_1, v_1, w_1) enter into A_{11}, B_{11}, C_{11}, while variations of the parameters enter into A_{12}, B_{12}, C_{12}. Forming, from the system (2)–(5) for the zeroth approximation and the system obtained for the first approximation, a bilinear combination and integrating it over the volume of the sphere, we obtain

$$\begin{aligned} I = &\int d\tau\, (u_1 A_0 - u_0 A_{11} + v_1 B_0 - v_0 B_{11} + w_1 C_0 - w_0 C_{11}) = \\ &= \chi_0^2 \overline{K} a^{-2} \int d\tau\, (u_0^2 + v_0^2 + w_0^2)\rho_1 + 2\overline{K} a^{-2} \chi_0 \chi_1 \int d\tau\, \rho_0 (u_0^2 + v_0^2 + w_0^2) + \\ &\quad + \int d\tau\, (A_{12} u_0 + B_{12} v_0 + C_{12} w_0). \end{aligned} \tag{13}$$

On the right-hand side of (13) stand quantities that may be regarded as known. Calculation of the left-hand side is somewhat cumbersome and as a result gives

$$\begin{aligned} I = &- \int d\tau\, \bar{\rho}_1 \left\{ \frac{\partial \psi_0}{\partial r} u_0 - \psi_0 \left[\frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (v_0 \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial w_0}{\partial \varphi} \right] \right\} \\ &+ \bar{K} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \sum_{i=1}^{n} \Big| r^2 \Big\{ K_1 u_0 \operatorname{div} \mathbf{u}_0 + N_1 \mu_1 \Big[u_0 \left(2 \frac{\partial u_0}{\partial r} - \frac{2}{3} \operatorname{div} \mathbf{u}_0 \right) \\ &+ v_0 (e_{r\theta})_0 + w_0 (e_{r\varphi})_0 \Big] \Big\} \Big|_{i-1}^{i}, \end{aligned} \tag{14}$$

where the summation extends over all layers of the model at whose boundaries the parameters of the problem have discontinuities, $|f|_{i-1}^{i} = f(r_i) - f(r_{i-1})$. Substituting (14) into (13), integrating over the angles, and passing to dimensionless variables, we obtain the required formula

$$
\begin{aligned}
\chi_1 = (2\chi_0 J)^{-1} \Bigg\{ & - \int_0^1 dx\, x^2 \rho_1 \left[(\chi_0^2 + 4vg_0 x^{-1}) z_{10}^2 - 2n(n+1) v g_0 x^{-1} z_{10} z_{30} \right. \\
& \qquad\qquad \left. + 2v z_{10} z_{60} + n(n+1)\chi_0^2 z_{30}^2 + \frac{2n(n+1)v}{x} z_{30} z_{50} \right] \\
& - 2Dv \int_0^1 dx\, x^2 \rho_0 z_{10} \left[2z_{10} - n(n+1) z_{30} \right] x^{-3} \int_0^x \rho_1 s^2 ds \\
& + \int_0^1 dx\, x^2 K_1 M_0^{-2} \left[4N_1 \mu_0 x^{-1} z_{10} + z_{20} - 2n(n+1) N_1 \mu_0 x^{-1} z_{30} \right]^2 \\
& + N_1 \int_0^1 dx\, \mu_1 M_0^{-2} \left[12K_0^2 z_{10}^2 - 8K_0 x z_{10} z_{20} - 12n(n+1) K_0^2 z_{10} z_{30} \right. \\
& \qquad\qquad + \frac{4}{3} x^2 z_{20}^2 + 4n(n+1) K_0 x z_{20} z_{30} \\
& \qquad\qquad + 2n(n+1) \left(n(n+1) \left(2K_0^2 + N K_0 \mu_0 + \frac{8}{9} N_1 \mu_0^2 \right) - M_0^2 \right) z_{30}^2 \\
& \qquad\qquad \left. + x^2 M_0^2 n(n+1) (N_1 \mu_0)^{-2} z_{40}^2 \right] \Bigg\},
\end{aligned}
\tag{15}
$$

where

$$
J = \int_0^1 dx\, x^2 \rho_0 \{ z_{10}^2 + n(n+1) z_{30}^2 \}. \tag{16}
$$

In fluid regions one should set $\mu_0 = \mu_1 = 0$. Formula (15) is the principal one. It makes it possible, on the basis of a single initial real Earth model, to consider the entire set of real models that are close to the initial one. In addition, (15) makes it possible to construct a theory of attenuation of spheroidal oscillations of the Earth. For $n = 0$ and $z_6 = 0$, (14) passes into the corresponding formula for radial oscillations [3], which was obtained by another method and was used to explain the anomalously weak attenuation of radial oscillations.

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REFERENCES

1. V. N. Zharkov, *Izv. AN SSSR, ser. geofiz.*, No. 2, 8 (1962).

2. V. N. Zharkov, V. M. Lyubimov et al., in the book *Earth Tides and the Internal Structure of the Earth*, Nauka, 1967; *Fizika Zemli*, No. 1, No. 2, No. 5 (1967).
3. V. N. Zharkov, V. M. Lyubimov, *DAN*, 177, No. 2 (1967).
4. *Natural Oscillations of the Earth*, Moscow, 1964.

Note: Figure translations are in progress. See original paper for figures.

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