ON RECURSIVE FUNCTIONS OF LARGE RANGE

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Abstract

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MATHEMATICS

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ON RECURSIVE FUNCTIONS OF LARGE RANGE

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Scollem in ([1]) showed that every general recursive function (g.r.f.) $f(x)$ can be represented in the form

$$f(x)=\psi(\mu y\{\varphi(y)=x\})=\psi(\varphi^{-1}(x)), \tag{1}$$

where ψ, φ are suitable primitive recursive functions (p.r.f.). Scollem further writes that, probably, a necessary and sufficient condition for ψ to be able to represent all g.r.f.' s in the form (1) with some p.r.f. $\varphi(x)$ is again that ψ be a function of large range, but that he did not investigate this. However, as E. A. Polyakov has observed, there is no single such function ψ by means of which it would be possible to represent any one-place g.r.f. in the form (1). Nevertheless Scollem' s conjecture proved to be valid for a representation of one-place g.r.f.' s somewhat different from (1).

Lemma 1. *For every p.r.f. of large range $R(x)$ there is a p.r.f. of large range $K(x)$ such that $R(x)=R(K(x))$.*

Let, for each $a \in N=\{0,1,2,\ldots\}$, all solutions of the equation $R(x)=a$ be written in the sequence

$$x_0^{(a)}<x_1^{(a)}<x_2^{(a)}<\cdots. \tag{*}$$

Then, if $x=x_j^{(a)}$, put $K(x)=x_i^{(a)}$, where $i=\mu_t\{\operatorname{rest}(j,P_t)=0\}$, $\operatorname{rest}(x,y)$ is the remainder on division of x by y, and P_t is the prime number with number t. Since every $x \in N$ belongs to one of the sequences of type (*), $K(x)$ will be an everywhere-defined p.r.f. of large range:

$$K(x)=\mu t_{\leq x}\{\,|(L_1(t)+1)\cdot \operatorname{sg}|R(t)-R(x)|-$$

$$-(\mu i_{\leq L_1(x)}\{\operatorname{rest}(L_1(x),P_i)=0\}+1)|=0\,\},$$

where $L_1(x)$ is a p.r.f. of large range which, together with $R(x)$, carries out a simple one-to-one enumeration of all pairs of natural numbers (see ([2]), p. 136).

Theorem 1. *If $R(x)$ is a p.r.f. of large range, then: 1) for every partial recursive function (p.r.f.) $f(x)$ defined at zero there exist a p.r.f. $F(x)$ and an integer a such that*

$$f(x) = R(F^{-1}(x)) + a \cdot \overline{\operatorname{sg}}\, x; \tag{2}$$

2) for every partial recursive function $f(x)$ which is not defined at zero, there exist a p.r.f. $F(x)$ and a partial recursive function

$$\rho(x) = \begin{cases} \text{undefined}, & \text{if } x = 0, \\ 0, & \text{if } x > 0, \end{cases}$$

such that

$$f(x) = R(F^{-1}(x)) + \rho(x). \tag{3}$$

Proof. Let $f(x)$ be an arbitrary partial recursive function. Then there is a p.r.f. $F_1(x, y)$ such that

$$f(x) = R(\mu y\{F_1(x, y) = 0\})$$

(see ([2]), p. 137). By Lemma 1, there exists a p.r.f. of large range $K(x)$ such that $R(x) = R(K(x))$. Take a p.r.f. $L_2(x)$, which together with $K(x)$ carries out a simple one-to-one enumeration of all pairs of natural numbers, and the function

$B(x) = \mu y\{F_1'(x, y) = 0\}$ can be written in the following way:

$$B(x) = K\left(\mu t\left\{L_2(t) \cdot \overline{\operatorname{sg}}\, F_1(L_2(t), K(t)) \cdot \prod_{i=0}^{K(t)-1} \operatorname{sg} F_1(L_2(t), i) = x\right\}\right) +$$

$$+B(0) \cdot \overline{\operatorname{sg}}\, x,$$

i.e.

$$B(x) = K(F^{-1}(x)) + B(0) \cdot \overline{\operatorname{sg}}\, x,$$

where

$$F(x) = L_2(x) \cdot \overline{\text{sg}}\, F_1(L_2(x), K(x)) \cdot \prod_{i=0}^{K(x)-1} \text{sg}\, F_1(L_2(x), i).$$

Then

$$f(x) = R(K(F^{-1}(x)) + B(0) \cdot \overline{\text{sg}}\, x) =$$

$$= R(K(F^{-1}(x))) + R(B(0) \cdot \overline{\text{sg}}\, x) - R(0),$$

$$f(x) = R(F^{-1}(x)) + (R(B(0)) - R(0)) \cdot \overline{\text{sg}}\, x.$$

If $f(0)$ is defined, then from (4) we obtain (2); if $f(0)$ is not defined, then we obtain (3).

It is also obvious that if, for some p.r.f. $R(x)$, all one-place g.r.f.' s are representable in the form (2), then $R(x)$ must be a function of large span.

Remark. Let M be the collection of all p.r.f.' s of large span which differ from one another only at zero. The following assertion is true: there exists such a set M that for every g.r.f. $f(x)$ there are p.r.f.' s $\psi(x) \in M$ and $\varphi(x)$ such that

$$f(x) = \psi(\varphi^{-1}(x)). \tag{5}$$

The representation (5) is a certain strengthening of the result in (1).

Next we consider the algebras $\mathfrak{A}_{\text{o.r.}} = \langle A_{\text{o.r.}}; +, *, -1 \rangle$ and $\mathfrak{A}_{\text{q.r.}} = \langle A_{\text{q.r.}}; +, *, -1 \rangle$ (see ([3])). Let

$$M(x) = a_0 + a_1 x + a_2 x^2 + ... + a_m x^m, \qquad P(x) = b_0 + b_1 x + b_2 x^2 + ... + b_n x^n$$

$(m, n > 1,\ a_0 \neq 0)$ be polynomials with rational coefficients such that

$$(\forall x)[x \in N \to M(x) \in N \ \&\ P(x) \in N].$$

Define the function $T_P(x)$ as the distance from x to the nearest number on the right of the form $P(t)$.

Theorem 2. The functions $M(x)$ and $T_P(x)$ form a basis of the algebra $\mathfrak{A}_{\text{q.r.}}$.

Proof. From the functions $M(x)$,

$$x \dot{-} y = T_P(T_P^{-1}(x) + y) = x - y \quad \text{for } x \geq y,$$

$$D_a(x) = \mu y\{ay = x\} = [x/a] \quad \text{for } x = a \cdot t,$$

with the aid of the operations $+, *, -1$ one can obtain all linear functions, x^2, $x \cdot y$, and a polynomial with integer coefficients $P_1(x) = a \cdot P(x)$, where a is a suitable number from N;

$$\overline{\mathrm{sg}}\, x = \begin{cases} 1, & x = 0, \\ 0, & x > 0, \end{cases} \qquad x \dot{-} y = \begin{cases} x - y, & x \geq y, \\ 0, & x < y. \end{cases}$$

Let

$$l_1(x) = \mu y\{P_1(y) = a \cdot (x + T_P(x))\},$$

i.e.

$$l_1(x) = P_1^{-1}(x) * a \cdot (x + T_P(x)).$$

We note that

$$(\exists x_0)(\forall x)[x > x_0 \ \&\ P(i) < x \leq P(i+1) \to l_1(x) = i + 1],$$

i.e. $l_1(x)$ assumes all natural-number values greater than d_0 ($P(d_0) \geq x_0$), and

$$(\exists t)(\forall x)[x > t \to P(x) \geq x^2].$$

One can verify that in the sequence

$$\langle l_1(0), T_P(0)\rangle,\ \langle l_1(1), T_P(1)\rangle,\ \langle l_1(2), T_P(2)\rangle, \ldots$$

there will occur any pair of natural numbers of the form (a^2, a), where $a > t$. Then for $x > t \vee x = 0$:

$$[\sqrt{x}] = T_P(\mu z\{(l_1(z) + 1) \cdot \mathrm{sg}(T_P^2(z) \dot{-} l_1(z)) = x + 1\}). \tag{6}$$

In order for formula (6) to be valid for all x, it is enough to make a small transformation. Now $q(x) = x \dot{-} [\sqrt{x}]^2$, and the functions $x + 1$ and $q(x)$ form a basis of the algebra $\mathfrak{A}_{\text{q.r.}}$ (see ([2]), p. 121).

Remark. Let the function $T_{at^n}(x)$ be equal to the distance from x to the nearest number on the right of the form $a \cdot t^n$ ($a > 0,\ n \geq 2$). Then the function $T_{at^n}(x)$ and an arbitrary q.r.f. $f(x)$, where $f(0) = d \neq 0$, form a basis of the algebra $\mathfrak{A}_{\text{q.r.}}$.

The proof of this assertion can be reduced to the proof of Theorem 3, after first obtaining from $f(x)$ and

$$T_{at^n}^n(x) = a\left(\left[\sqrt[n]{\frac{x}{a}}\right] + \mathrm{sg}\, x\right)^n - x$$

the function $x^{\tilde{n}}$ ($\tilde{n} > 1$).

For p.r.f. an analogous result was obtained by I. A. Lavrov ([4]).

It is not hard to see that a basis of the algebra $\mathfrak{A}_{\text{o.r.}}$ is also a basis of the algebra $\mathfrak{A}_{\text{p.r.}}$. The question arises: will every o.r. basis of the algebra $\mathfrak{A}_{\text{p.r.}}$ be a basis of the algebra $\mathfrak{A}_{\text{o.r.}}$ (by a basis is meant a system of generating elements of the algebra)?

Theorem 3. *There exists an o.r. basis of* $\mathfrak{A}_{\text{p.r.}}$ *which is not a basis of* $\mathfrak{A}_{\text{o.r.}}$.

Proof. From the remark to Theorem 3 it follows, for example, that $x+1$ and $T_{2t^2}(x)$ form a basis of the algebra $\mathfrak{A}_{\text{p.r.}}$. But these functions belong to a proper subalgebra of the algebra $\mathfrak{A}_{\text{o.r.}}$ (see ([5])) and therefore cannot be a basis of $\mathfrak{A}_{\text{o.r.}}$.

Next we construct examples of partial recursive functions of large span, each of which together with $x+1$ does not form a basis of the algebra $\mathfrak{A}_{\text{p.r.}}$; this will give a negative answer to one of A. I. Mal’ tsev’ s questions.

Let $\Phi_i(x)$ be equal to the distance from x to the nearest, on the left (or on the right), number of the form i^t, $\Phi_i(0)=0$ $(i=2,3,...)$. Define the function

$$Q_i(x)=\begin{cases}\text{undefined}, & \Phi_i^{-1}(\Phi_i(x))=x,\\ \Phi_i(x) & \text{otherwise};\end{cases}$$

$Q_i(x)$ is undefined at those points x at which $\Phi_i(x)$ assumes, for the first time, one of its values. It is clear that $Q_i(x)$ is a p.r.f. of large span, and there is an interval of arbitrarily large length in which it is nowhere defined.

Theorem 4. *The function* $x+1$ *and the p.r.f. of large span* $Q_i(x)$ *are not a basis of* $\mathfrak{A}_{\text{p.r.}}$.

Proof. It is enough to note that the functions $x+1$ and $Q_i(x)$ belong to the following set C, closed under $+$, $*$, and -1:

$$f(x)\in C \Longleftrightarrow \left\{(\exists a)(\exists b)\left[\begin{array}{l} a,b \text{ rational},\\ a>0,\end{array} \quad \& \, f(x)=\begin{cases} ax+b, & x\in D_f,\\ \text{undefined} & \\ \text{otherwise}, & \end{cases}\right]\vee\right.$$

$$\left.\vee\,(\forall m)(\forall n)\left[m,n\in N\&m>0\&f(mx+n)\begin{array}{c}\text{is undefined in}\\ \text{infinitely many points}\end{array}\right]\right\};$$

where D_f is the domain of definition of f.

The question of whether every o.r.f. of large span, together with the function $x+1$, forms a basis of $\mathfrak{A}_{\text{p.r.}}$, remains open.

In the paper ([6]) R. Robinson showed that from the functions $x+1$,

$$\chi(x)=\begin{cases}1, & x\neq t^2,\\ 0, & x=t^2\end{cases}$$

with the aid of the operations $|x-y|$, $*$, and i, one can obtain all unary p.r.f. Let us consider the analogous question for o.r.f.

Theorem 5. *The functions* 1 *and* $[\sqrt{x}]$ *form a basis of the algebra*

$$\mathfrak{A}'_{\text{o.r.}} = \langle A_{\text{o.r.}};\ |x-y|,\ *,\ -1\rangle.$$

Let us note that if in $\mathfrak{A}_{\text{p.r.}}$ the operation $+$ is replaced by the operation $|x-y|$, then the basis is simplified.

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Note: Figure translations are in progress. See original paper for figures.