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Abstract

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ON EIGENVECTORS OF THE HAMMERSTEIN OPERATOR

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In this paper we consider the problem of the existence of eigenvectors of the Hammerstein operator

$$Ax(t) = \int_\Omega k(t,s) f[s, x(s)]\, ds. \tag{1}$$

Here Ω is a bounded closed set in a finite-dimensional space. In what follows it is assumed that the function $f(s,u)$, defined on $\Omega \times R^n$ with values in R^n, satisfies the Carathéodory conditions and is potential ($f(s,u) = \operatorname{grad} G(s,u)$; $G(s,0) = 0$); the matrix $k(t,s)$ $(t, s \in \Omega)$ is symmetric, measurable with respect to the aggregate of variables, and the linear operator defined by it

$$Kx(t) = \int_\Omega k(t,s) x(s)\, ds \tag{2}$$

in the Hilbert space of vector-functions $H = \mathcal{L}_2$ is self-adjoint and has no more than a finite number of negative eigenvalues (each of finite multiplicity).

For an operator A acting in some real Banach space E, a vector $x_0 \in E$ $(x_0 \neq 0)$ is called an eigenvector if $Ax_0 = \lambda_0 x_0$, where λ_0 is some number (an eigenvalue of the operator A corresponding to the eigenvector x_0).

In the problem of the existence of eigenvectors of operator (1), the variational method has received the greatest development; its beginning was laid by the works of Hammerstein, Golomb, and Lichtenstein. In Golomb' s works a general theoretical-functional scheme for solving this problem was indicated, and the case was considered in which the superposition operator $fx(s) = f[s, x(s)]$ acts in $\mathcal{L}_2$, which means "sublinearity" of the function $f(s,u)$ in u. M. M. Vainberg was the first to consider the case in which the operator K has eigenvalues of different signs. The next important step was the work of M. A. Krasnosel' skii, in which the operator f acts in various spaces $\mathcal{L}_p$ (but the basic functional of the variational problem is defined on $\mathcal{L}_2$), i.e., functions $f(s,u)$ with power growth in u are considered. Further results were obtained by M. M. Vainberg, M. A. Krasnosel' skii, Ya. B. Rutitskii, I. V. Shragin, and others (see [1–6], and the detailed bibliography in [1–3]). Recently one of the authors [7,8] has carried out a detailed analysis of the properties of operators in general Banach spaces of measurable functions (the so-called ideal spaces). The use of the results of these works in the problem of eigenvectors of operator (1) makes it possible, on the one hand, to considerably expand the class of nonlinearities under study, and, on the other hand, to dispense with requirements usually imposed on the operators K and f (for example, the complete continuity of K or the continuity of f, which is essential already in the passage to Orlicz spaces). In the present note we present results obtained along these lines.

1. A Banach space E of measurable almost everywhere finite vector-functions on Ω with values in R^n is called **ideal** if from

$|x| \leqslant |y|$, where $y \in E$, and x is measurable on Ω, it follows that $x \in E$ and $\|x\|_E \leqslant \|y\|_E$ (by $|x|$ is denoted the vector whose components are equal to the moduli of the components of x; inequalities for vectors are understood componentwise).

The space E of vector-functions may be regarded as the direct sum of n spaces $E_1, \dots, E_n$ of scalar functions. Let Ω_i $(i = 1, \dots, n)$ be the supports of E_i, i.e. such subsets of Ω that every function in E_i vanishes outside Ω_i, while in E_i there exist functions that are positive for almost all $s \in \Omega_i$. The space E' **dual** to E is the space of vector-functions whose components vanish outside the supports Ω_i of the spaces E_i and for which the norm

$$\|y\|_{E'} = \sup_{\|x\|_E \leqslant 1} \int_\Omega (x(s), y(s))\, ds \tag{3}$$

is meaningful.

Among ideal spaces are the space E_{u_0} (u_0 is a nonnegative measurable function) of vector-functions for which the norm

$$\|x\|_{E_{u_0}} = \inf\{\lambda :\ |x| \leqslant \lambda u_0\}, \tag{4}$$

is meaningful, the space E'_{u_0} dual to it, the spaces $\mathcal{L}_p$, Orlicz spaces, and many others.

A set in E is called w-**bounded** if, for every $\varepsilon > 0$, this set has an ε-net N such that $|x| \leqslant u_0$ for all $x \in N$ and for some $u_0 \in E$. An operator acting from one ideal space E_1 into another E_2 is called w-bounded if it takes every norm-bounded set into a w-bounded set. Examples of w-bounded sets are sets of functions with equicontinuous norms. One says that a linear operator K, acting from E_1 into E_2, has the **Ando σ-property** if

$$\lim_{\text{mes}\, D\to 0} \|P_{DKP}D\|_{E_1\to E_2} = 0, \tag{5}$$

where P_D is the operator of multiplication by the characteristic function of the set $D \subseteq \Omega$.

Let H_1 be the linear span of the eigenvectors of operator (2) in H corresponding to negative eigenvalues. Denote $I = -P_1 + P_2$, where P_1 and P_2 are the projection operators, respectively, onto H_1 and $H_2 = H \ominus H_1$.

2. Everywhere below it is assumed that the operator f acts from E into E', and the linear operator K (and therefore $\widetilde{K} = IK$) acts from E' into E, where E is some ideal space, E' is the space dual to it, and moreover $E \subseteq H$. By virtue of a theorem of M. A. Krasnosel' skii–S. G. Krein [7], the operator $\widetilde{K}^{1/2}$ in this case acts from E' into H and simultaneously from H into E. It is clear that the operator $G[s, x(s)]$ acts from E into $\mathcal{L}_1$, and therefore on H the Golomb functional is defined

$$\Phi(y) = \int_\Omega G[s, \widetilde{K}^{1/2}y(s)]\, ds. \tag{6}$$

This functional (without additional assumptions) turns out to be differentiable on H; its gradient is the operator $\widetilde{K}^{1/2}f\widetilde{K}^{1/2}$, acting in H. The eigenvectors of the Hammerstein operator $A = Kf$ correspond to critical points of the functional $\Phi(y)$ on the spheres $(y, y) = c^2$ (in the case when the operator K is positive definite in H) or on the hyperboloids $(Iy, y) = c^2$ (in the case when the operator K has a finite number of negative eigenvalues).

3. Suppose that the operator K is positive definite in H. In this case $\widetilde{K} = K$ and $\widetilde{K}^{1/2} = K^{1/2}$.

Theorem 1. *Let one of the following conditions be satisfied: a) K is completely continuous as an operator from E' into E; b) there exists such a nonnegative func-*

function $M(u)$, $M(u)/u \to \infty$ as $u \to \infty$, such that the operator $M\{G[c, x(s)]\}$ acts from E into $\mathcal{L}_1$; c) $E = E_{u_0}$.

Then the Hammerstein operator $A = Kf$ has in E a continuum of semiradially representable eigenvectors $x = K^{1/2}y$ in every ellipsoid $(y, y) \le c^2$ $(0 < c < \infty)$.

If $f(s, u) \ne 0$ for $u \ne 0$, then eigenvectors exist on every ellipsoid $(y, y) = c^2$. Moreover, if $(u, f(s, u)) > 0$ for $u \ne 0$, then the eigenvalues corresponding to these eigenvectors are positive.

The proof is based on the fact that, under the conditions of the theorem, the Golomb functional $\Phi(y)$ turns out to be weakly continuous in H.

Theorem 2. Let $f(s,-u) = -f(s,u)$ and $(u, f(s,u)) > 0$ for $u \neq 0$. Suppose one of the following conditions is fulfilled: a) K is completely continuous as an operator from E' into E, and the operator f is continuous; b) the operator K has Ando' s σ-property; c) the operator f is w-bounded; d) $E = E_{u_0}$.

Then the Hammerstein operator $A = Kf$ has at least a countable number of distinct semiradially representable eigenvectors $x = K^{1/2}y$ on every ellipsoid $(y,y) = c^2$. The only limit point of the corresponding eigenvalues is 0.

Under the conditions of this theorem the functional $\Phi(y)$ in H turns out to be weakly continuous, smooth, even, and nonnegative. Therefore, for the proof one may use the theorem of L. A. Lyusternik–M. A. Krasnosel' skii (see (2)).

4. Let now the operator K have a finite number of negative eigenvalues. In this case we shall additionally assume that the function $G(s,u)$ satisfies the inequality

$$G(s,u) \geq (qu,u) + Q(s,u), \tag{7}$$

where q is a symmetric positive definite matrix, and the function $Q(s,u)$ satisfies the Carathéodory conditions and defines the superposition operator $Qx(s) = Q[s,x(s)]$, acting from E into $\mathcal{L}_1$ and asymptotically zero-quadratic:

$$\lim_{\|x\|_E \to \infty} \|Qx\|_{\mathcal{L}_1} / \|x\|_E^2 = 0. \tag{8}$$

Sufficient conditions for the fulfillment of the last equality are indicated in (9). Inequality (7) means that the Golomb functional satisfies the growth condition

$$\lim_{\|y\|_H \to \infty,\ (Iy,y)>0} \Phi(y) = +\infty. \tag{9}$$

Theorem 3. Let one of the following conditions be fulfilled: a) K is completely continuous as an operator from E' into E; b) there exists a nonnegative function $M(u)$, $M(u)/u \to \infty$ as $u \to \infty$, such that the operator $M\{G[s,x(s)]\}$ acts from E into $\mathcal{L}_1$ and is bounded; c) $E = E_{u_0}$.

Then the Hammerstein operator $A = Kf$ has a continuum of semiradially representable eigenvectors $x = K^{1/2}y$ in every hyperboloid $(Iy,y) \geq c^2$ $(0 < c < \infty)$.

If $f(s,u) \neq 0$ for $u \neq 0$, then eigenvectors exist on every hyperboloid $(Iy,y) = c^2$. Moreover, if $(u, f(s,u)) > 0$ for $u \neq 0$, then the eigenvalues corresponding to these eigenvectors are positive.

It is clear that among the eigenvectors of the operator $A = Kf$ under the conditions of the theorem there are vectors of arbitrarily large norm. If, moreover, the Golomb functional is nonnegative and vanishes only at 0, then there also exist eigenvectors of arbitrarily small norm.

The proof of the theorem is based on the fact that under its conditions the functional $\Phi(y)$ turns out to be weakly lower semicontinuous. Other conditions can also be indicated for the fulfillment of the latter property, not connected with the -

assumption (7). Thus, with the aid of Fatou' s lemma it is not difficult to show that the Golomb functional is weakly lower (upper) semicontinuous if $G(s,u) \geq G_1(s,u)$ $(G(s,u) \leq G_1(s,u))$, where $G_1(s,u)$ satisfies the Carathéodory conditions and generates a weakly continuous Golomb functional (in particular, if the gradient of the latter is compact). The last assertion contains Lemma 3 from (9).

Theorem 4. *Suppose the conditions of Theorem 2 are satisfied. Then the Hammerstein operator $A = Kf$ has no fewer than $2m$ distinct semiconically representable eigenvectors $x = K^{1/2}y$ on each hyperboloid $(Iy, y) = c^2$, where m is the dimension of the space H_1.*

The proof, as was already noted, is based on the fact that under the conditions of the theorem the Golomb functional is weakly continuous and smooth. In addition, it is taken into account that the genus of the hyperboloid $(Iy, y) = c^2$ is equal to m.

5. In conclusion we note that the results of the article extend in a natural way to operators with Lebesgue integral with respect to an arbitrary measure, in particular to infinite systems; moreover, instead of Banach spaces one may consider locally convex spaces.

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Note: Figure translations are in progress. See original paper for figures.