ON DIFFERENCE APPROXIMATIONS OF DIFFERENTIAL OPERATORS OF MATHEMATICAL PHYSICS

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Abstract

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ON DIFFERENCE APPROXIMATIONS OF DIFFERENTIAL OPERATORS OF MATHEMATICAL PHYSICS

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We shall consider linear equations of mathematical physics on an n-dimensional Riemannian manifold M, which are written with the aid of invariant operators d and δ in the form

$$L(d,\delta)\omega = f, \tag{1}$$

where L is a matrix of polynomials in d and δ, and ω and f are vectors of differential forms on M (f is given, and ω is sought) (see, for example, ($^{1-6}$)). In this form one can write, for example:

a) the Laplace (Laplace-Beltrami) equation $\Delta u = f$ ($\delta d\omega = f$, where ω and f are 0-forms); if the metric is

$$ds^2 = \sum_{i=1}^{n} dx_i^2, \quad \text{then} \quad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2};$$

b) the biharmonic equation $\Delta^2 u = f$ ($\delta d\,\delta d\,u = f$);

c) the equations of the theory of elasticity in Lamé form

$$\operatorname{grad}\operatorname{div} u - \frac{1-2\sigma}{2(1-\sigma)}\operatorname{rot}\operatorname{rot} u = f$$

$$\left(d\delta + \frac{1-2\sigma}{2(1-\sigma)}\,\delta d\right)\omega = \varphi,$$

where ω and φ are 2-forms;

d) the basic system of vector analysis

$$\operatorname{rot} u = a, \quad \operatorname{div} u = b, \quad d\omega = \alpha, \quad \delta\omega = \beta,$$

where ω is a 2-form, α a 3-form, β a 1-form;

e) the stationary system of Maxwell equations—example d) or $d\omega = 0$, $\delta\omega = \varphi$, where ω is a 1-form and φ a 0-form, etc.

Combinatorial topology provides a natural apparatus for constructing difference schemes on an arbitrary cellular decomposition for system (1). An important property of the proposed schemes is the presence in them of the "conservation laws" inherent in (1).

By a **mesh** $K(M)$ on the manifold M we shall mean a finite oriented polyhedral cell complex K for which $|K| = |M|$ (i.e., a polyhedral cellular decomposition of M). Denote by Ω^p the space of p-forms on M, and by Ω_p the space of p-dimensional currents on M (see ([1,2])); by $C_p(K)$ and $C^p(K)$, respectively, the spaces of p-dimensional chains and cochains on K (all over the field of real numbers) (see ([7])).

There are two obvious mappings: $\psi_* : C_p(K) \to \Omega_p$, assigning to each chain from $C_p(K)$ the current from Ω_p with the same carrier, and $\psi : \Omega^p \to C^p(K)$, assigning to a form ω the cochain $\psi\omega$ such that

$$(\psi\omega, e) = \int_{\psi_* e} \omega.$$

Obviously, $\psi d = d_h \psi^*$.

* There exists a right inverse to ψ, a mapping $\varphi : C^p \to \Omega^p$, $\psi\varphi = 1_{C^p}$, such that $\varphi(\sigma) = 0$ near $M - \operatorname{St}\sigma$, $\varphi d_h^* = d\varphi$, and $\varphi 1 = 1$ (see, for example, ([2]), pp. 196–197).

Let, further, $K_h(M)$ be a sequence of meshes on M that "refine" as $h \to 0$. Suppose also that in the spaces $C^p(K_h)$ a Euclidean structure $\langle , \rangle$ has been introduced, approximating the Euclidean structure in $\Omega^p(M)$: $\langle \psi_h\omega, \psi_h\omega \rangle \to \langle \omega, \omega \rangle$ as $h \to 0$.

Our recipe for writing a difference scheme for equation (1) on an arbitrary mesh K_h consists in the following. The unknown is a vector-cochain ω_h, satisfying the equation

$$L(d_h, \delta_h)\omega_h = f_h, \tag{1_h}$$

where d_h is the coboundary operator on $C^*(K_h)$, $\delta_h \equiv d_h^*$ is its adjoint in the metric on $C^*(K_h)$, approximating the metric on Ω^*, and f_h is a vector-cochain approximating the form f, for example $\psi_h f$. We shall not formulate conditions for convergence of the solutions ω_h to ω as $h \to 0$; let us note, however, that in the case where M is a closed manifold (without boundary), and also in the case of certain "classical" boundary conditions, equations a)–d) admit a quadratic "energy integral."

The difference equations constructed according to our recipe admit all the same integrals. From the existence of the latter there usually follows convergence of the solutions of the difference equations to the solutions of the differential ones. Finally, we point out the obvious connection of our recipe with A. A. Dorodnitsyn's method of integral relations (^{8}). Let us consider several examples. (Below ${}^p\omega$ everywhere denotes a p-cochain.)

1. **A regular square mesh with step $h = 1$ in the plane** (see Fig. 1). Introduce a metric on cochains by the following formulas:

$$\langle {}^0\omega, {}^0\omega\rangle = \sum_i {}^0\omega_i^2; \qquad \langle {}^1\omega, {}^1\omega\rangle = \sum_i {}^1\omega_i^2; \qquad \langle {}^2\omega, {}^2\omega\rangle = \sum_i {}^2\omega_i^2,$$

where the summation extends over all vertices, edges, and faces. (Recall that the number ${}^1\omega_1$ is defined on the edge with number 1, and ${}^2\omega_1$ on the face with number 1—see Fig. 1.) The Cauchy-Riemann system $u_x - v_y = \alpha$, $u_y + v_x = \beta$ is approximated on such a mesh in the following way: one seeks a 1-cochain ${}^1\omega$ satisfying the equations

$$d_h {}^1\omega = {}^2\alpha, \qquad \delta_h {}^1\omega = {}^0\beta,$$

where

$${}^2(d_h {}^1\omega)_1 = {}^1\omega_1 + {}^1\omega_2 - {}^1\omega_3 - {}^1\omega_4; \qquad {}^0(\delta_h {}^1\omega)_0 = {}^1\omega_1 + {}^1\omega_4 - {}^1\omega_5 - {}^1\omega_6$$

(if one uses the functions u and v, this corresponds to a "checkerboard" mesh (see Fig. 2): u is given in the "circles," and v in the "crosses" ; the equations are written "at the points" marked by "squares"). If, for example, $\alpha = 0$, then ${}^1\omega = d_h {}^0\varphi$, and we have $\Delta_h {}^0\varphi = {}^0\beta$, where $\Delta_h = \delta_h d_h$ is the usual difference approximation of the scalar Laplace operator.

2. **An orthogonal curvilinear mesh in the plane with linear element** $ds^2 = (h_1 dx_1)^2 + (h_2 dx_2)^2$ (see also (^{9})). Consider a decomposition into squares with step $h = 1$ along the axes x_1, x_2. The metric on cochains is introduced in the following way:

$$\langle {}^0\omega, {}^0\omega\rangle = \sum_i {}^0(h_1 h_2)_i \, {}^0\omega_i^2, \qquad \langle {}^1\omega, {}^1\omega\rangle = \sum_i {}^1(h_1 h_2)_i \, {}^1\omega_i^2,$$

$$\langle {}^2\omega, {}^2\omega\rangle = \sum_i {}^2(h_1 h_2)_i^2 \, \omega_i^2.$$

Here ${}^{j}(h_1 h_2)_i$ ($j = 0, 1, 2$) denote the value of the area at the corresponding element of the decomposition on which the quantity ${}^{j}\omega_i$ is defined. Geometrically the operators d_h and δ_h are similar to those defined in example 1 (more precisely, d_h coincide, while δ_h differ by coefficients). For computing δ_h

the following natural device is applied: let $\tilde{\omega}(i)$ be a basis j-cochain concentrated on the i-th j-dimensional basis element. Then

$$\langle \delta_h \omega, \tilde{\omega}(i) \rangle = (\delta_h \tilde{\omega})_i = \langle \omega, d\tilde{\omega}(i) \rangle,$$

where the right-hand side is known.

Fig. 1

Fig. 2

3. A regular parallelogram grid on the plane with steps h_i along the axes x_i and acute angle α ($\cos\alpha = \theta$, $\sin\alpha = \theta'$)—see Fig. 3. The metric on 0-cochains and 2-cochains is introduced as "orthogonal," i.e.

$$\langle {}^{0}\omega, {}^{0}\omega \rangle = h_1 h_2 \theta' \sum_i {}^{0}\omega_i^2,$$

$$\langle {}^{2}\omega, {}^{2}\omega \rangle = \sum_i {}^{2}\omega_i^2,$$

and on 1-cochains it is oblique-angular:

$$\langle {}^{1}\omega, {}^{1}\omega \rangle = \frac{h_1 h_2 \theta'}{2} \sum_i \left(\sum_{j=1}^{4} \left(\frac{{}^{1}\omega_j}{h^j} \right)^2 - 2\theta \sum_{j=1}^{4} \frac{\pm {}^{1}\omega_j {}^{1}\omega_{j+1}}{h_1 h_2} \right),$$

Fig. 3

where the summation over i extends over all parallelograms, and over j—over the sides of the i-th parallelogram; h^j is equal to h_1 or h_2 depending on the direction of the j-th edge, and the sign ± in the second term is determined depending on whether the angle between the j-th and $(j+1)$-st edges is acute or obtuse; the 5-th edge in this sum is, obviously, simply the 1-st. Using the device indicated in Example 2, we compute the operator on 1-cochains (see Fig. 3):

$$(\delta_h {}^{1}\omega)_0 = \left(-\frac{{}^{1}\omega}{h_1} - \frac{{}^{1}\omega_1}{h_2} + \frac{{}^{1}\omega_7}{h_1} + \frac{{}^{1}\omega_{10}}{h_2} \right) -$$

$$-\frac{\theta}{8} \left(\frac{{}^{1}\omega_8}{h_2} + \frac{{}^{1}\omega_6}{h_2} + \frac{{}^{1}\omega_{11}}{h_1} + \frac{{}^{1}\omega_9}{h_1} - \frac{{}^{1}\omega_2}{h_2} - \frac{{}^{1}\omega_3}{h_1} - \frac{{}^{1}\omega_5}{h_1} - \frac{{}^{1}\omega_{12}}{h_2} \right).$$

If ${}^1\omega = d_h {}^0\varphi$, then $\delta_h {}^1\omega = \Delta_h {}^0\varphi$, which gives, for $h_1 = h_2 = 1$,

$$(\Delta_h {}^0\varphi)_0 = {}^0\varphi_1 + {}^0\varphi_3 + {}^0\varphi_5 + {}^0\varphi_7 - 4{}^0\varphi_0 + \theta({}^0\varphi_4 + {}^0\varphi_8 - {}^0\varphi_2 - {}^0\varphi_6).$$

4. A regular cubic grid with step $h = 1$ in three-dimensional space. The natural metric on cochains is analogous to that considered in Example 1. A difference approximation of system d) leads us to a scheme of the type proposed by V. I. Lebedev (10) and written directly for u.*

The above-mentioned scheme with the octahedral grid at first glance gives the impression of being ingenious, but somewhat artificial; we hope that our description shows its complete naturalness.

5. Operators on a triangulation generally cannot be constructed on the basis of the coordinate form of equations; in our technique this case presents no difficulty.

Remark 1. It is known that in many equations of mathematical physics, besides the operators d_h and δ_h, there occurs the metric operator $*$.

We have not succeeded in constructing a satisfactory local analogue of this operator on finite complexes. Moreover, in the present note we have not touched upon questions connected with multiplication of cochains. These questions require further study.

Remark 2. The domain of applicability of the formal apparatus of elementary combinatorial topology in mathematical physics is not limited to questions of approximation of differential equations. This apparatus is an analytical apparatus for certain problems of mathematical physics that are discrete in essence.

Let us consider the following examples:

a) See (12,13). Kirchhoff's equations for the distribution of current in a system of conductors may be formulated as follows: on a two-dimensional complex with a metric on the edges proportional to the resistances, find a 1-chain $V = RJ$ (whose values on the edges are the products $R_i J_i$) satisfying the equations

$$d_{hV} = E, \qquad \delta_{hV} = 0,$$

where E is the 2-chain of electromotive forces (its value on a face is equal to the sum of the e.m.f.' s along the boundary of the face). Integrating the second equation of the system, we can introduce the loop current ψ: $V = \delta\psi$, which satisfies the Poisson equation $\Delta_h \psi = E$, where $\Delta_h = d_h \delta_h$ is the Laplace operator on 2-chains of the complex. This last form shows the solvability of the system and indicates possible methods of solving it. Analogous considerations are admitted by the potential case $d_{hV} = 0$, $\delta_{hV} = Q$ with current sources.

* In (11) the results of (10) are extended to very general classes of equations.

b) The equations of a hydraulic system are

$$\delta_{hG} = 0, \qquad d_{hG} = E,$$

where G is a certain 1-chain (the flow-rate function), and the metric (the resistances R_i of example a)) itself depends on G. Analogously to example a), the first equation admits integration—the introduction of loop flows ψ. The equation $d_h \delta_h \psi = E$ in this case is already nonlinear; however, the existence of a solution follows from variational considerations (the existence of an energy $\langle \delta_h \psi, \delta_h \psi \rangle = \langle \psi, E \rangle$).

c) The problem of the equilibrium of a system of crossed threads reduces to the following: find a 0-chain u (the deflection of the threads from the plane at the point of fastening) satisfying the equilibrium equation

$$\Delta_{hu} = f, \qquad \Delta_h = \delta_h d_h,$$

where f is a 0-chain—the load acting at the fastening points of the threads—and the metric depends only on the lengths of the threads. On the boundary the position φ of the threads may be prescribed in one form or another.

Remark 3. In the present note we have deliberately touched only on local questions. The usefulness of applying the topological apparatus in the study of global questions (existence and number of solutions) is obvious. For example, the completeness of the Kirchhoff system and the introduction of loop currents are questions easily resolved in terms of the homology of the complex of conductors (see, for example, ([13])), etc.

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Note: Figure translations are in progress. See original paper for figures.

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