ON EXTENSIONS, CHARACTERISTIC FUNCTIONS, AND GENERALIZED RESOLVENTS OF SYMMETRIC OPERATORS

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Abstract

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MATHEMATICS

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ON EXTENSIONS, CHARACTERISTIC FUNCTIONS, AND GENERALIZED RESOLVENTS OF SYMMETRIC OPERATORS

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1. Let A be a closed symmetric operator in a Hilbert space H. The domain D_A of the operator A is not assumed to be dense in H. For an arbitrary nonreal z set $\mathfrak{N}_z = H \ominus (A - zE)D_A$. As is known ([1]), D_A and $\mathfrak{N}_z$ are linearly independent, and D_A, $\mathfrak{N}_z$, and $\mathfrak{N}_{\bar{z}}$ are linearly independent if and only if $\overline{D}_A = H$. Put $\mathfrak{P}_z = \mathfrak{N}_z \cap (D_A + \mathfrak{N}_{\bar{z}})$. $\mathfrak{P}_z = \{0\}$ if and only if $\overline{D}_A = H$. In the Cartesian product $\mathfrak{N}_z \times \mathfrak{N}_{\bar{z}}$ we distinguish the linear manifold

$$\mathfrak{G}_z = \{[\psi, \varphi] : \varphi - \psi \in D_A\}.$$

$\mathfrak{G}_z$ is the graph of an isometric operator X_z, with domain $\mathfrak{P}_z$ and range $\mathfrak{P}_{\bar{z}}$ (([1]), Theorem 9). It is clear that $X_z^{-1} = X_{\bar{z}}$. In a somewhat different way the operator X_z was defined in ([2]), where it plays an essential role in the construction of symmetric extensions of the operator A.

In the present paper the operator X_z is used in describing dissipative and other extensions of the operator A. A formula is established connecting X_z with the characteristic function of the operator A. Using these terms, we describe the totality of all generalized resolvents of the operator A.

2. A linear operator B in H is called **dissipative** if, for every $f \in D_B$, $\mathrm{Im}(Bf, f) \geq 0$ (cf. ([3])). Such an operator B is called **maximal dissipative** if it has no proper dissipative extensions in H. We shall call a linear operator B in H **accumulative** if $\mathrm{Im}(Bf, f) \leq 0$ $(f \in D_B)$; analogously to the preceding, we define the notion of a **maximal accumulative operator**. The domain D_B of any closed maximal dissipative or accumulative operator B is dense in H ([3]).

Denote by $\mathfrak{K}_z$ $(\mathrm{Im}\, z \neq 0)$ the class of all linear nonexpanding operators F from $\mathfrak{N}_z$ into $\mathfrak{N}_{\bar{z}}$ $(D_F \subset \mathfrak{N}_z)$, and by $\mathfrak{F}_z$ the totality of all $F \in \mathfrak{K}_z$ for which $D_F = \mathfrak{N}_z$. We shall call a linear operator F from $\mathfrak{N}_z$ into $\mathfrak{N}_{\bar{z}}$ **admissible** if $F\psi = X_z\psi$ only for $\psi = 0$. If the operator $F \in \mathfrak{F}_z$ is admissible, then the adjoint operator $F^* \in \mathfrak{F}_{\bar{z}}$ is also admissible.

Theorem 1. *For any nonreal z in the lower (upper) half-plane, the formulas*

$$D_B = D_A + (F - E)D_F, \tag{1}$$

$$B(f + F\psi - \psi) = Af + zF\psi - \bar{z}\psi \quad (f \in D_A,\ \psi \in D_F) \tag{2}$$

establish a one-to-one correspondence between the totality of all admissible operators $F \in \mathfrak{K}_z$ and the totality of all dissipative (accumulative) extensions B of the operator A. Moreover,

$$F = (B - \bar{z}E)(B - zE)^{-1}\big|_{\mathfrak{N}_z \cap (B - zE)D_B}.$$

The operator B is closed and maximal if and only if it corre-

corresponds to the admissible operator $F \in \mathfrak{F}_z$. In this case the operator B^*, adjoint to B, corresponds (with z replaced by $\bar{z}$) to the operator $F^* \in \mathfrak{F}_{\bar{z}}$.

We shall say that the linear operators S and T in H are formally adjoint if, for all $f \in D_S$ and $g \in D_T$,

$$(Sf, g) = (f, Tg).$$

Theorem 2. For every nonreal z there exists a one-to-one correspondence between the totality of all extensions B of the operator A, formally adjoint to A, and the totality of all linear manifolds $\mathfrak{L} \subset \mathfrak{N}_z \times \mathfrak{N}_{\bar{z}}$ satisfying the condition $[\psi, X_z\psi] \in \mathfrak{L}$ only when $\psi = 0$. This correspondence is defined by the formulas

$$D_B = D_A \dotplus \{\varphi - \psi :\ [\psi, \varphi] \in \mathfrak{L}\},$$

$$B(f + \varphi - \psi) = Af + z\varphi - \bar{z}\psi \qquad (f \in D_A,\ [\psi, \varphi] \in \mathfrak{L}),$$

which is equivalent to the equality

$$\mathfrak{L} = \{[\psi, \varphi] \in \mathfrak{N}_z \times \mathfrak{N}_{\bar{z}} :\ \varphi - \psi \in D_B,\ B(\varphi - \psi) = z\varphi - \bar{z}\psi\}.$$

The operator B is closed if and only if the corresponding linear manifold $\mathfrak{L}$ is closed.**

3. Denote by A_λ ($\operatorname{Im}\lambda \neq 0$) the extension of the operator A, defined on the linear manifold $D_{A_\lambda} = D_A \dotplus \mathfrak{N}_z$ by the formula

$$A_\lambda(f + \varphi) = Af + \lambda\varphi \qquad (f \in D_A,\ \varphi \in \mathfrak{N}_\lambda).$$

According to Theorem 1, A_λ for $\operatorname{Im}\lambda > 0$ ($\operatorname{Im}\lambda < 0$) is a closed maximal dissipative (accumulative) extension of the operator A, and $A_\lambda^* = A_{\bar{\lambda}}$. Fix an arbitrary nonreal λ_0 and denote by Π the open upper or lower half-plane containing λ_0. By the same Theorem 1, for every $\lambda \in \Pi$ the operator A_λ corresponds to the admissible operator $C(\lambda) \in \mathfrak{F}_{\bar\lambda}$, defined by the formula

$$C(\lambda) = (A_\lambda - \lambda_0 E)(A_\lambda - \bar\lambda_0 E)^{-1}\,\big|_{\mathfrak{N}_{\bar\lambda}},$$

The operator function $C(\lambda)$ $(\lambda \in \Pi)$ is called the **characteristic function of the operator** A.*** It depends analytically on λ, and

$$\|C(\lambda)\| \le \left|\frac{\lambda - \lambda_0}{\lambda - \bar\lambda_0}\right| \qquad (\lambda \in \Pi).$$

Put

$$\Pi_\varepsilon = \{\lambda \in \Pi :\ \varepsilon < |\arg\lambda| < \pi - \varepsilon\} \quad (0 < \varepsilon < \pi/2).$$

Let $\Phi(\lambda)$ $(\lambda \in \Pi)$ be an arbitrary analytic operator function whose values are linear nonexpanding operators mapping one Hilbert space $\mathfrak{N}$ into another $\mathfrak{N}'$. Put, then,

$$\Omega_\Phi = \left\{h \in \mathfrak{N} :\ \lim_{\lambda\to\infty,\ \lambda\in\Pi_\varepsilon} |\lambda|(\|h\| - \|\Phi(\lambda)h\|) < \infty\right\},$$

and denote by $\Phi_0(\infty)$ the operator with domain of definition Ω_Φ, given by the formula

$$\Phi_0(\infty)h = \lim_{\lambda\to\infty,\ \lambda\in\Pi_\varepsilon} \Phi(\lambda)h \qquad (h \in \Omega_\Phi). \tag{3}$$

According to the results of the work ([9]), Ω_Φ is a linear manifold and for every $h \in \Omega_\Phi$ the strong limit (3) exists.

* This theorem intersects with some results of the papers ([2–5]) and, in particular, with Theorem 1.1.1 of ([3]).

** Theorem 2 is related to the results of the paper ([6]).

*** The concept of the characteristic function of a linear operator was first introduced by M. S. Livšic ([7]) for isometric and densely defined symmetric operators with defect index $(1,1)$ and their extensions. Subsequently this concept was generalized and modified in various ways. In the sense of the definition proposed in ([8]), the operator function $C(\lambda)$ $(\lambda \in \Pi)$ considered here is the characteristic function of the operator A_{λ_0}.

Theorem 3. The formulas

$$\Phi_{\lambda_0} = \Omega_C, \qquad X_{\lambda_0} = C_0(\infty)$$

hold.

From this, in particular, it follows that

Theorem 4*. The domain of definition of the operator A is dense in H if and only if, for every nonzero $\varphi \in \mathfrak{N}$,

$$\lim_{\lambda\to\infty,\ \lambda\in\Pi_\varepsilon} [|\lambda|(\|\varphi\| - \|C(\lambda)\varphi\|)] = \infty.$$

4. We shall agree to denote by A_F the operator $B \supset A$ corresponding to the admissible operator $F' \in \mathfrak{F}_{\lambda_0}$ according to formulas (1), (2) for $z = \lambda_0$. In an analogous sense the notation A_{F^*} will be used, so that $(A_F)^* = A_{F^*}$.

Recall that the operator-valued function R_λ ($\operatorname{Im}\lambda \neq 0$) is a generalized resolvent of the operator A if and only if it can be represented in the form $R_\lambda = P(\widetilde{A} - \lambda E)^{-1}|H$, where $\widetilde{A}$ is some self-adjoint extension of the operator A with exit into a Hilbert space $\widetilde{H} \supset H$, and P is the orthoprojector in $\widetilde{H}$ onto H.**

Theorem 5. The formula

$$R_\lambda = \begin{cases} (A_{F(\lambda)} - \lambda E)^{-1} & (\lambda \in \Pi), \\ (A_{F^*(\bar{\lambda})} - \lambda E)^{-1} & (\bar{\lambda} \in \Pi) \end{cases} \tag{4}$$

establishes a one-to-one correspondence between the set of all generalized resolvents R_λ ($\operatorname{Im}\lambda \neq 0$) of the operator A and the set of all analytic operator-valued functions $F(\lambda)$ ($\lambda \in \Pi$) with values in $\mathfrak{F}_{\lambda_0}$, satisfying the condition

$$F_0(\infty)\psi = X_{\lambda_0}\psi \quad \text{only for } \psi = 0. \tag{5}$$

This condition is equivalent to the following:

$$\lim_{\lambda\to\infty,\ \lambda\in\Pi_\varepsilon} \left| \frac{1}{\lambda}([E - C(\lambda)F(\lambda)]^{-1}\psi, \psi) \right| = 0 \quad \text{for every } \psi \in \mathfrak{N}_{\lambda_0}. *** \tag{6}$$

Theorem 6. If $F(\lambda)$ ($\lambda \in \Pi$) is an analytic operator-valued function with values in $\mathfrak{F}_{\lambda_0}$ satisfying condition (5) or (6), then

$$\lim_{\lambda\to\infty,\ \lambda\in\Pi_\varepsilon} \left\{ \frac{1}{\lambda}[E - C(\lambda)F(\lambda)]^{-1} \right\} = 0$$

in the sense of strong convergence.

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CITED LITERATURE

[1] M. A. Naimark, *Izv. AN SSSR, Ser. Mat.*, **4**, No. 1, 53 (1940).
[2] M. A. Krasnosel' skii, *DAN*, **59**, No. 1, 13 (1948).
[3] R. S. Phillips, *Sborn. per. Matematika*, **6**, 4, 11 (1962).
[4] A. V. Shtraus, *Izv. AN SSSR, Ser. Mat.*, **18**, No. 1, 51 (1954).
[5] B. I. Lomkarev, *Tr. II nauch. konf. matem. kafedr pedagog. inst. Povolzh' ya*, vol. 1, Kuibyshev, 1962, p. 66.
[6] A. I. Plesner, *DAN*, **66**, No. 4, 557 (1949).

[7] M. S. Livshits, *Matem. sborn.*, **19** (61), 2, 239 (1946).
[8] A. V. Shtraus, *Izv. AN SSSR, Ser. Mat.*, **24**, No. 1, 43 (1960).
[9] A. V. Shtraus, *Izv. AN SSSR, Ser. Mat.*, **30**, No. 6, 1325 (1966).
[10] A. V. Shtraus, *DAN*, **67**, No. 4, 611 (1949).
[11] M. A. Naimark, *Izv. AN SSSR, Ser. Mat.*, **4**, No. 3, 277 (1940).
[12] M. A. Naimark, *Izv. AN SSSR, Ser. Mat.*, **7**, No. 6, 285 (1943).

* Cf. ([10]), Theorem 5. In the case of an operator with defect index $(1, 1)$, the theorem 4 obtained here is close to one result of M. S. Livshits ([7]), Theorem 15).

** The notions of the spectral function and generalized resolvent of a densely defined symmetric operator were first defined by M. A. Naimark ([11,12]). In the case of a non-densely defined closed symmetric operator these notions are extended in ([4]).

*** If $\overline{D_A} = H$, then conditions (5), (6) are trivial. For this case formula (4) was established in ([4]). There the theorem on the properties characterizing generalized resolvents of a non-densely defined operator was also proved. On its basis, B. I. Lomkarev ([5]) extended formula (4) to the case of an operator A with non-dense domain of definition; however, the additional condition imposed on $F(\lambda)$ had a rather complicated form.

Note: Figure translations are in progress. See original paper for figures.