ON SOME STRUCTURAL PROPERTIES OF SUBSEMIGROUPS IN A GROUP

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ON SOME STRUCTURAL PROPERTIES OF SUBSEMIGROUPS IN A GROUP

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This note is a continuation of the author’ s works on the theory of subsemigroups in a group (1–3), etc. Systematic investigations of the theory of subsemigroups in a group were begun in the works of P. G. Kontorovich (4,5) and of P. G. Kontorovich and the author (6). The main result of the note is the proof of necessary and sufficient conditions, in terms of the structure of subsemigroups $P(G)$ of a torsion-free group G, for a subsemigroup P of the torsion-free group G to be a pure (i.e., without pairs of mutually inverse elements) invariant subsemigroup in its group closure. Here, as always, the empty set is regarded as a subsemigroup of every subsemigroup.

For brevity of exposition and convenience of reference we use the notions of an S-product and an S-element, as well as some terms and notation from (7), although we note that with equal success one could have used the notion of a special product (8) or of a maximal element (9). In addition, instead of an S-product one could introduce and use a shorter condition.*

In the note we use generally accepted notation; only to denote an S-element (and an S-product) do we use the letter F (and f), since P denotes a pure invariant subsemigroup (which, generally speaking, may be regarded as a positive cone, and it is customary to denote it by P (10)). Lower-case letters denote elements of the group G, upper-case letters denote elements of the structure $P(G)$ and the subsemigroups of the group G coinciding with them.

Let us give the definitions of an S-product and an S-element. An element $f \in \{x, y\}$ will be called an S-product of the elements x and y if the subsemigroup $\{x, y\}$ contains more than two elements and if, for $z \in \{x, y\}$ and $z \neq f$, from $f \in \{x, y\}$ it follows that $z = y$, and from $f \in \{z, y\}$ it follows that $z = x$.

Let Σ be a structure and $X, Y \in \Sigma$. An S-element for X and Y will mean an element $F \in \Sigma$ satisfying the following conditions: 1) $F \in (X, Y)$; 2) if Z is a one-covering element of (X, Y) distinct from F, then from $F \in (X, Y)$ it follows that $Z = Y$, and from $F \in (Z, Y)$ it follows that $Z = X$. Here and below $(X, Y, \dots, W)$ denotes the principal ideal in Σ generated by the union of the elements $X, Y, \dots, W$, i.e., the set of all elements of Σ that are less than the union $X \vee Y \vee \cdots \vee W$ or equal to it. An element T of the structure Σ is called one-covering if the ideal (T) has a true (i.e., distinct from T) greatest element ([11]).

If the product xy of elements x and y of the subsemigroup Γ is not representable in the form of any word in the alphabet x, y distinct from xy and yx, then the product xy will be called single-valued ([3]).

We shall use the following known facts:

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* An element $f \in \{x, y\}$ will be called a C-product of the elements x and y if the subsemigroup $\{x, y\}$ contains more than two elements and if, for $u, v \in \{x, y\}$ and $u \neq f \neq v$, from $f \in \{u, v\}$ it follows that $u = x, v = y$ or $u = y, v = x$. Analogously one can introduce a C-element.

Lemma 1. If the product xy is single-valued, then it is an S-product.*

Lemma 2. If f is an S-product of the elements x, y, then at least one of the equalities $f = xy,\ f = yx$ holds.**

Lemma 3. Let x, y be elements of a group G without torsion. Then, in order that an element $f \in G$ be an S-product of the elements x and y, it is necessary and sufficient that in $P(G)$ the element $\{f\}$ be an S-element for the elements $\{x\}$ and $\{y\}$.***

A singly covering element in the structure $P(G)$ of a group G is a cyclic subsemigroup $((^9), Lemma 2.11)$. In the structure $P(G)$ of a torsion-free group G the converse assertion also holds: every cyclic subsemigroup is a singly covering element $((^9), Lemma 2.10)$. We shall often use the indicated assertions.

We shall also use the necessary and sufficient conditions, formulated in the language of the structure $P(G)$ of subsemigroups of a group G, for the subsemigroup $\{a, b\}$ of some group G to be a noncyclic commutative semigroup all of whose elements have infinite order (Lemma 6 of ([7])). We denote these conditions by $(*)$ (we do not give them for lack of space).

Lemma 4. For singly covering elements X, Y of the structure $P(G)$ there exist no more than two singly covering S-elements.

Lemma 5. Let the semigroup Γ satisfy the following three conditions:

1. The cancellation law holds in Γ.
2. $\Gamma x = x\Gamma$ for all elements $x \in \Gamma$.
3. The semigroup Γ has an externally adjoined identity; i.e., $xy = e$ implies $x = y = e$.

Then the product xy of any two elements $x \neq e \neq y$ of the semigroup Γ is single-valued.

Corollary. The product of any two elements x, y of a semigroup Γ satisfying the conditions of Lemma 5 is an S-product of these elements.

Theorem. Let P be a subsemigroup in a torsion-free group G. In order that the subsemigroup P be purely invariant in its group closure H as a subsemigroup, it is necessary and sufficient that the ideal (P) and the structure $P(G)$ satisfy the following conditions:

1. For any two singly covering elements X, Y from the ideal (P) there exists at least one singly covering S-element, but no more than two distinct singly covering S-elements.
2. If a singly covering S-element for singly covering elements X, Y from the ideal (P) is an atom, then each of the elements X, Y is an atom.
3. If singly covering elements A, S from the ideal (P) have only one singly covering S-element F and the ideal (A, S) is distinct both from (A) and from (S) and the conditions $(*)$ are not fulfilled, then there exists a singly covering element $S_1 \neq S,\ S_1 \in P$, such that one of the singly covering S-elements for the elements A, S will coincide with F.
4. If singly covering elements A, S from the ideal (P) have only one singly covering S-element, then each singly covering element from the ideal (A, S) that is not contained in the ideals (A), (S) and is not an S-element for the elements A, S, is an S-element for singly covering elements A, S_2, where $S_2 \neq S,\ S_2 \in (P)$.
5. If there exist two distinct singly covering S-elements F_1, F_2 for singly covering elements A, S from the ideal (P), then for each singly covering S-element F_1, F_2 there is found such a singly covering element S_i $(S_i \neq S,\ i = 1, 2)$ from the ideal (P), such that one of the singly covered S-elements F_3, F_4 for the elements A, S_i will coincide with any prescribed singly covered S-element F_1 or F_2.

Proof. We give the proof of necessity only for item 3. Let P be a pure subsemigroup, invariant in its group closure, in the torsion-free group G. Suppose the singly covered elements A and S have only one singly covered S-element F, the ideal (A, S) is distinct both from (A) and from (S), and the conditions $(*)$ are not satisfied. Since the elements A, S, and F are singly covered, $A = \{a\}$, $S = \{s\}$, and $F = \{f\}$. By Lemma 3, the element f is an S-product of the elements a and s, and, by Lemma 2, $f = as$ or $f = sa$. Let $f = as$; the case $f = sa$ is proved analogously. Since the conditions $(*)$ are not satisfied, the semigroup $\{a, s\}$ is either cyclic, or noncommutative, or has torsion. However, the semigroup $\{a, s\}$ cannot be cyclic, since the elements a and s belong to the pure semigroup P and the ideal (A, S) is distinct from (A) and (S). Moreover, all elements of the semigroup $\{a, s\}$ have infinite order, since the group G is

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* Lemma 1 from (^{7}), see also Lemma 4.5 from (^{9}).
** Lemma 2 from (^{7}), see also Lemma 4.4 from (^{9}).
*** This lemma is a direct consequence of Lemma 3 from (^{7}).

torsion-free. Therefore, from the fact that the conditions $(*)$ are not satisfied it follows that the semigroup $\{a, s\}$ is noncommutative, i.e. $as \neq sa$. By invariance of the semigroup P, $as = s_1 a$, where $s_1 \neq s$, $s_1 \in P$. The product $s_1 a = f_1$ is an S-product of the elements s_1 and a by the corollary to Lemma 5. By Lemma 3, the element $F_1 = \{f_1\}$ will be an S-element for the elements $A = \{a\}$ and $S_1 = \{s_1\}$. It is clear that $F = F_1$ and $S_1 \neq S$.

Sufficiency. From item 2 it follows that the semigroup P is pure. We show that the semigroup P is invariant in its group closure. For this it is enough to show that for any elements $a, s \in P$ there exist elements $s_1, s_2 \in P$ such that the two equalities hold: 1) $as = s_1 a$ and 2) $sa = as_2$. By item 1, for arbitrary singly covered elements A and S from the ideal (P), there exists at least one singly covered S-element, but not more than two distinct singly covered S-elements. We consider two possibilities.

a) Suppose there exists exactly one singly covered S-element F for the elements A and S from the ideal (P). The elements F, A, S are singly covered, and therefore $F = \{f\}$, $A = \{a\}$, $S = \{s\}$ $(a, s \in P)$. By Lemma 3, the element f is an S-product of the elements a and s. By Lemma 2, $f = as$ or $f = sa$. Suppose, for example, that $f = as$. We show that in this case equality 1) $as = s_1 a$, where $s_1 \in P$, will hold. If $as = sa$, then there is nothing to prove. If the ideal (A, S) coincides with one of the ideals (A) or (S), then $\{a, s\} = \{a\}$ or $\{a, s\} = \{s\}$ and, consequently, either $s = a^n$, or $a = s^m$. In both cases $as = sa$, and there is nothing to prove. Suppose now that the ideal (A, S) is distinct both from (A) and from (S) and suppose $as \neq sa$, i.e. the conditions $(*)$ are not satisfied. Then, by item 3, there is a singly covered element $S_1 \neq S$, $S_1 \in (P)$, such that one of the singly covered S-elements for the elements $A = \{a\}$ and $S_1 = \{s_1\}$, for example F_1, will coincide with F (a singly covered S-element for the elements A and S_1 exists, by item 1, and $F_1 = \{f_1\}$). By Lemma 3, the element f_1 is an S-product of the elements a and $s_1 \in P$, and, by Lemma 2, $f_1 = as_1$ or $f_1 = s_1 a$. The equality $f_1 = as_1$ is impossible. Indeed, $F_1 = F$, whence it follows that $f_1 = f$, by assumption $f = as$, therefore $as_1 = f_1 = f = as$ and $s_1 = s$, which contradicts the condition $S_1 \neq S$. Therefore $f_1 = s_1 a$, and we have $as = f = f_1 = s_1 a$, where $s_1 \in P$. We now prove the second equality: $sa = as_2$, $s_2 \in P$. The elements a and s, as before, are assumed noncommuting. Then the element sa is not an S-product of the elements a and s, for otherwise, by Lemma 3, the element $\{sa\}$ would be a singly covered S-element for the elements A and S, and $\{sa\} \neq \{as\} = F$, which contradicts the assumption that the elements A and S have only one singly covered S-element. By Lemma 3, the element $\{sa\}$ is not an S-element for the elements A and S. Further, $\{sa\} \subset \{a, s\}$, therefore the element $\{sa\} \in (A, S)$, but the element $\{sa\} \in (A)$ and

$\{sa\} \in (S)$, since from $\{sa\} \in A$ it follows that $\{sa\} \subset \{a\}$ and $sa = a^n$, whence the elements a and s commute, which is false. Likewise, $\{sa\} \in (S)$ entails the commutativity of the elements a and s. Now, by item 4, the element

$\{sa\}$ is an S-element for the one-covering elements A and $S_2 = \{s_2\}$, where $S_2 \neq S,\ S_2 \in (P)$. Consequently, by Lemma 3, the element sa is an S-product of the elements a and $s_2 \in P$, and by Lemma 2, either $sa = s_2a$ or $sa = as_2$. The equality $sa = s_2a$ is impossible, since it entails the equality $s = s_2$ and $S = S_2$, which is false. Thus it remains that $sa = as_2$, where $s_2 \in P$. In the case where $f = sa$, the proofs of equalities 1) and 2) are carried out analogously.

b) Suppose there exist two S-elements F_1 and F_2 $(F_1 \neq F_2)$ for arbitrary one-covering elements A and S from the ideal P. The elements $A,\ S,\ F_1$, and F_2 are one-covering, and therefore $A = \{a\}, S = \{s\}, F_1 = \{f_1\}$, and $F_2 = \{f_2\}$. By Lemma 3, the elements f_1 and f_2 are S-products of the elements a and s. By Lemma 2, $f_1 = as$ or $f_1 = sa$, and $f_2 = as$ or $f_2 = sa$. From $F_1 \neq F_2$ it follows that $f_1 \neq f_2$. Therefore either $f_1 = as,\ f_2 = sa$, or $f_1 = sa,\ f_2 = as$. In both cases the proof is carried out in the same way. Let, for example, $f_1 = as,\ f_2 = sa$. First let us show that in this case equality 1) $as = s_1a$ will be valid, where $s_1 \in P$. If $as = sa$, then there is nothing to prove. Therefore we shall assume further that the elements a and s do not commute. By item 5, for the S-element F_1 there is a one-covering element $S_1 \neq S$ such that one of the one-covering S-elements for the elements $A = \{a\}$ and $S_1 = \{s_1\}$ (where $s_1 \neq s$), for example F_3, will coincide with $F_1,\ F_3 = F_1$ (an S-element for the elements A and S_1 exists by item 1, and $F_3 = \{f_3\}$). By Lemma 3, the element f_3 is an S-product of the elements a and s_1, and by Lemma 2, $f_3 = as_1$ or $f_3 = s_1a$. The equality $f_3 = as_1$ is impossible. Indeed, $F_3 = F_1$ and $f_3 = f_1$, but, by assumption, $f_1 = as$, therefore $as_1 = f_3 = f_1 = as$ and $s_1 = s$, which is false. Therefore $f_3 = s_1a$, and we have $as = f_1 = f_3 = s_1a$. From the equality $f_2 = sa$, by analogous reasoning we obtain $sa = f_2 = f_4 = as_2$. Thus both equalities 1) and 2) have been proved.

Let us note that, on the basis of the theorem proved, the author has obtained necessary and sufficient conditions, in terms of the structure of subsemigroups of an arbitrary semigroup, for a semigroup to be a directed group, analogously to how this is done for an ordered group in paper (^{12}).

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Note: Figure translations are in progress. See original paper for figures.