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Abstract

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ON THE ADMISSIBLE ORDER OF GROWTH OF THE CHARACTERISTIC OF QUASICONFORMALITY IN THE THEOREM OF M. A. LAVRENT' EV

(Presented by Academician M. A. Lavrent' ev on 14 XI 1967)

In our paper ([1]) the following assertion of M. A. Lavrent' ev was proved:

Every locally homeomorphic quasiconformal mapping $f : E^n \to E^n$ of Euclidean space E^n of dimension $n \geqslant 3$ into itself is automatically a homeomorphism onto the whole space E^n.

Here we shall refine this result by showing to what extent the condition of quasiconformality of the mapping f can be weakened.

Recall that a mapping $f : D \to D'$ of a domain $D \subset E^n$ onto a domain $D' \subset E^n$ is called K-quasiconformal in D $(1 \leqslant K < \infty)$ if the quantity

$$\delta(x) = \lim_{r\to 0} \frac{\max_{|\zeta-x|=r} |f(\zeta) - f(x)|}{\min_{|\zeta-x|=r} |f(\zeta) - f(x)|} \tag{1}$$

is bounded in D, and moreover

$$\delta(x) \leqslant K \tag{2}$$

for almost all points $x \in D$.

A mapping is called quasiconformal in D if it is K-quasiconformal for some K $(1 \leqslant K < \infty)$.

In the case when the K-quasiconformal mapping f is homeomorphic in D, it is known that between the modulus of any family of curves Γ from D and the modulus of its image $\Gamma' \subset D'$ the relation ([3]) holds

$$K^{-(n-1)} M(\Gamma) \leqslant M(\Gamma') \leqslant K^{n-1} M(\Gamma). \tag{3}$$

Let now f be an arbitrary mapping of the domain D; let a be some fixed point in D; $B(a,r) = B(r)$ is the open ball of radius r with center a.

Consider the quantity

$$K(r) = K(a,r) = \sup_{x \in D \cap B(r)} \delta(x). \tag{4}$$

We wish to characterize the maximum admissible rate of growth of $K(r)$ as $r \to \infty$ for the case $D = E^n$, under which the assertion of M. A. Lavrent' ev' s theorem still remains valid.

Let us note that in most qualitative questions of the theory of quasiconformal mappings, the necessary information extracted from relation (3) consists only in the fact that the inequalities

$$0 < M(\Gamma) < \infty,$$

$$0 < M(\Gamma') < \infty \tag{5}$$

always hold simultaneously. But for the simultaneous validity of relations (5) there is no need to impose on the mapping the condition of quasiconformality necessarily. This circumstance will be used to answer the main question of the present note.

Lemma. Let D be an unbounded domain in E^n; let Γ be a family of curves in D, each of which, as a set, is unbounded in E^n. Let, further, f be a homeomorphism of D taking the family Γ into a family Γ' whose modulus is positive. Then

$$\int^{\infty} \frac{dr}{rK(r)} < \infty,$$

where $K(r)$ is the quantity determined by relation (4).

First of all, note that the function $K(r)$ is nondecreasing; therefore, in the case when, for some r_0, $0 \le r_0 < \infty$, $K(r_0) = \infty$, the assertion of the lemma is trivial. Thus we may assume that the function $K(r)$ is finite everywhere.

Consider the sequence of domains $D_m = D \cap A_m$ $(m = 1, 2, ...)$, where $A_m = \{x \mid m < |x-a| < m+1\}$ is a spherical ring. Suppose first that all curves of the family Γ meet some sphere $S_{m_0} = \{x \mid |x-a| = m_0\}$. Then, by virtue of the conditions of the lemma, on each curve $\gamma \in \Gamma$ one can select a sequence of simple arcs γ_m $(m = m_0, m_0+1, m_0+2, ...)$, each of which lies in the corresponding domain D_m and joins the boundary spheres of the ring A_m. Thus there naturally arises a sequence of families $\Gamma_m = \bigcup_{\gamma \in \Gamma} \gamma_m$ $(m = m_0, m_0+1, ...)$.

The curves of the family Γ_m form part of all possible curves lying in the ring A_m and joining its boundary spheres; therefore, by the first Grötzsch principle ([2],[3]),

$$M(\Gamma_m) \leq \omega_{n-1}\left[\ln\frac{m+1}{m}\right]^{1-n}, \qquad m = m_0,\ m_0+1, \ldots, \tag{6}$$

where ω_{n-1} is the area of the $(n-1)$-dimensional unit sphere.

By construction, the families Γ_m lie in nonintersecting domains D_m, and each of them is minorizing with respect to the original family Γ, i.e., for every curve $\gamma \in \Gamma$ there is a curve $\gamma_m \in \Gamma_m$ such that $\gamma_m \subset \gamma$. Hence the images Γ'_m and Γ' of the families under consideration are in the same relation. By virtue of the second Grötzsch principle ([2],[3]), we have

$$M^{1/(1-n)}(\Gamma') \geq \sum_{m=m_0}^{\infty} M^{1/(1-n)}(\Gamma'_m). \tag{7}$$

Since in $D \cap B(m)$ the mapping f is $K(m)$-quasiconformal, for any $m = m_0 + 1, m_0 + 2, \ldots$ we have

$$M(\Gamma'_{m-1}) \leq K^{n-1}(m) M(\Gamma_{m-1}),$$

and, taking (6) into account, we obtain

$$M(\Gamma'_{m-1}) \leq K^{n-1}(m)\omega_{n-1}\left[\ln\left(1+\frac{1}{m-1}\right)\right]^{1-n}, \qquad m = m_0+1,\ m_0+2, \ldots$$

Using these relations in (7), we find:

$$M^{1/(1-n)}(\Gamma') \geq \frac{1}{\omega_{n-1}} \sum_{m=m_0+1}^{\infty} \frac{\ln\left(1+\frac{1}{m-1}\right)}{K(m)}.$$

By the hypothesis of the lemma, $M(\Gamma') > 0$; therefore the series on the right-hand side of the last inequality must be convergent, which is equivalent to the convergence at infinity of the integral

$$\int^{\infty} \frac{dr}{rK(r)}. \tag{8}$$

To complete the proof of the lemma, it remains to dispense with the assumption made in the course of the proof that every curve $\gamma \in \Gamma$

intersects S_{m_0}. For this, let us decompose the original family Γ into a sequence of families L_m, defined as follows: the family L_m consists of those and only those curves of the family Γ which intersect the sphere $S_m = \{x \mid |x-a| = m\}$ and, for $m > 1$, do not intersect the preceding sphere S_{m-1}. Now consider the families L'_m, which are the images of the families of the sequence just constructed. Since $M(\Gamma') > 0$ and $\Gamma' = \bigcup_{m=1}^{\infty} L'_m$, at least one of the families L'_m, denote it by L'_{m_0}, must also have positive modulus ([2],[3]). If the preimage L_{m_0} of this family is taken as the original family Γ, then we are in the conditions of the special case already considered. This completes the proof of the lemma.

Thus, if Γ is the family of curves considered in the lemma, Γ' is its image under the mapping f, and

$$\int^{\infty} \frac{dr}{rK(r)} = \infty,$$

then $M(\Gamma) = 0$.

But, as is clear from the proof of M. A. Lavrent' ev' s theorem ([1]), its assertion remains valid for any locally homeomorphic mapping for which the image of the described family Γ has zero modulus; therefore the following theorem holds.

Theorem. *If $f : E^n \to E^n$ is a locally homeomorphic mapping of n-dimensional $(n \geqslant 3)$ Euclidean space E^n into itself, and the function $K(r)$ (the coefficient of quasiconformality of f in the ball of radius r with center at some fixed point $a \in E^n$) is such that $\int^{\infty} \frac{dr}{rK(r)}$ diverges, then f is a homeomorphism and, moreover, onto the whole space E^n.*

In the sense of the admissible order of growth of $K(r)$, this result is final. More precisely, we shall show that, whatever nondecreasing function $\varphi(r) \geqslant 1$ may be, for which

$$\int^{\infty} \frac{dr}{r\varphi(r)} < \infty,$$

one can construct a homeomorphic mapping f of the space E^n $(n \geqslant 2)$ onto the unit ball, the coefficient of quasiconformality $K(r)$ of which in the ball $|x| < r$, for all sufficiently large r, will be less than $\varphi(r)$.

We define the mapping f by the following formulas (vector notation):

$$f(x) = \begin{cases} \dfrac{1}{2}\dfrac{x}{c}, & \text{for } |x| \leqslant c, \\ \dfrac{1}{2}\left[1 + \left(\displaystyle\int_c^{\infty} \frac{dt}{t\varphi(t)}\right)^{-1} \displaystyle\int_c^{|x|} \frac{dt}{t\varphi(t)}\right] \dfrac{x}{|x|}, & \text{for } |x| > c, \end{cases}$$

where c is a positive constant, which we shall fix somewhat later.

It is easy to compute that for the mapping $f(x)$

$$K(r) = \left[\int_c^\infty \frac{dt}{t\varphi(t)} + \int_c^r \frac{dt}{t\varphi(t)}\right]\varphi(r)$$

for all sufficiently large values of r.

Now choosing the constant c so that

$$\int_c^\infty \frac{dt}{t\varphi(t)} < \frac{1}{2},$$

we obtain the desired mapping.

Thus, the condition of divergence of the integral (8) is the best possible sufficient condition on the whole in the class of locally homeomorphic mappings for the image of the space E^n to be the whole space E^n. However, from the example constructed it does not yet follow that this condition is the best sufficient condition guaranteeing the homeomorphy of the mapping in the entire space being mapped. Nevertheless, this example is easy to extend, obtaining under the same assumptions a mapping $\tilde{f}$, locally homeomorphic but not homeomorphic in E^n, which maps the whole space onto its proper part.

For this it is now enough to fix any locally homeomorphic, but not homeomorphic, quasiconformal mapping f_1 of the unit ball. If Q is the quasiconformality constant of the mapping f_1, then, obviously,

$$\widetilde{K}(r) \leq QK(r),$$

where $\widetilde{K}(r)$ is the coefficient of quasiconformality of the mapping $\tilde{f} = f_1 f$ in the ball $|x| < r$. Choosing now the constant c in the definition of the mapping f so that

$$\int_c^\infty \frac{dt}{t\varphi(t)} < \frac{1}{2Q},$$

we obtain $\widetilde{K}(r) < \varphi(r)$ for all sufficiently large values of r.

Let us note in conclusion that the integral (8) has already occurred ([4]) in connection with the proof of the impossibility of mapping the whole space E^3 homeomorphically and quasiconformally onto its proper part. The example constructed above shows that the admissible order of growth of the coefficient of quasiconformality obtained in ([4]) is sharp.

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Note: Figure translations are in progress. See original paper for figures.