GENERALIZED BOGOLIUBOV FUNCTIONS AND ELEMENTS OF THE CAUSAL \(S\)-MATRIX

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Abstract

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GENERALIZED BOGOLIUBOV FUNCTIONS AND ELEMENTS OF THE CAUSAL S-MATRIX

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In all investigations carried out within the framework of the axiomatic S-matrix approach to quantum field theory by N. N. Bogoliubov, B. V. Medvedev, M. K. Polivanov ([1,2]) and others ([3]), the principal quantities determining the theory are taken to be the class of so-called chronological operators

$$\begin{aligned}T_n^c(x_1,\dots,x_n) = T'(J(x_1)\dots J(x_n)) + \\ + \sum_{\substack{2\le m\le n-1\\ (\sum_{j=1}^m \nu_j = n,\ \nu_j\ge 1)}} \frac{(-i)^{n-m}}{m!} P(x_1,\dots,x_{\nu_1} \mid x_{\nu_1+1},\dots,x_{\nu_1+\nu_2} \mid \dots x_n)\times \\ \times T(\Lambda_{\nu_1}(x_1,\dots,x_{\nu_1})\Lambda_{\nu_2}(x_{\nu_1+1},\dots,x_{\nu_1+\nu_2})\dots\Lambda_{\nu_m}(\dots x_n)) + \\ + i^{n-1}\Lambda_n(x_1,\dots,x_n), \qquad n = 1,2,\dots;\end{aligned} \tag{1}$$

$$\Lambda_1(x_1) = J(x_1) = i\,\frac{\delta S}{\delta\varphi(x)}\,S^+; \tag{2}$$

$$\Lambda_\nu(x_1,\dots,x_\nu), \qquad \nu = 2,3,\dots,$$

are arbitrary timelike operators possessing the properties of Poincaré invariance, locality, symmetry, and local commutativity. In addition to these linear properties they must satisfy the equation of motion (see ([1])).

However, from the mathematical point of view the most convenient quantities for constructing the theory are not the chronological operators $T_n^c(x_1,\dots,x_n)$, $n = 1,2,\dots$, defined according to (1), but the infinite set of their vacuum averages, i.e.

$$B_n(x_1,\dots,x_n) = \langle 0|T_n^c(x_1,\dots,x_n)|\rangle, \qquad n = 1,2,\dots, \tag{3}$$

which serve as coefficient functions in the functional expansion of the S-matrix in asymptotic fields, i.e.

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int\dots\int\left(\prod_{j=1}^{n}d^4x_j\right)B_n(x_1,\dots,x_n):\varphi_{\text{out}}(x_1)\dots\varphi_{\text{out}}(x_n):. \quad (4)$$

The content of the BMP axioms (4,5) can be completely translated into the language of the vacuum averages $B_n(x_1,\dots,x_n)$, $n=1,2,\dots$, defined according to (3), characterized by linear and nonlinear relations.

A. Linear properties.

1°. Functional structures. The vacuum averages $B_n(x_1,\dots,x_n)$, $n=1,2,\dots$, are generalized functions: a) belonging to the space $S'(R^{4n})$; b) extendable to the space $S_+^{\text{KG}}(R^{4n})$ of sufficiently smooth solutions of the Klein-Gordon (KG) equation with positive energies, i.e. to the space $S_+^{\text{KG}}(R^{4n})$ of functions $f(x_1,\dots,x_n)$ representable in the form $f(x_1,\dots,x_n)=f_1(x_1)\dots$

$\dots f_n(x_n)$, where

$$f_j(x_j)=\frac{1}{(2\pi)^{3/2}}\int d^4p_j\,\theta(p_j^0)\,\delta(p_j^2-\mu^2)\,\tilde{f}_j(p_j)\,e^{ip_jx_j}\in S_+^{\text{KT}}(R^4), \quad (5)$$

$$\tilde{f}_j(p_j)\in S(\overline{\Omega}_\mu^+),\qquad j=1,2,\dots,n.$$

By $S(\overline{\Omega}_\mu^+)$ is denoted the space of test functions with
$\operatorname{supp}\tilde{f}_j(p_j)\in\overline{\Omega}_\mu^+$, where

$$\overline{\Omega}_\mu^+=\{p_j\in\overline{R}:\ (p_j^0)^2-(\mathbf{p}_j)^2=\mu^2,\quad p_j^0>0\},$$

$j=1,2,\dots,n.$

Condition b), in other words, means the existence of generalized functions $\widehat{B}_n(x_1,\dots,x_n)\in S_+^{\prime\text{KT}}(R^{4n})$, $n=1,2,\dots$, such that

$$B_n(f)=\widehat{B}_n(f)\quad\text{for any } f\in S_+^{\text{KT}}(R^{4n}). \quad (6)$$

Property b) requires the extendability (preextendability) of the generalized functions
$B_n(x_1,\dots,x_n)\in S'(R^{4n})$, $n=1,2,\dots$, to the mass surface, i.e. the existence of generalized M-functions defined by

$$M_n(p_1,\dots,p_n)=\prod_{j=1}^{n}\theta(p_j^0)\,\delta(p_j^2-\mu^2)\,\widetilde{B}_n(p_1,\dots,p_n)\in S'(\overline{\Omega}_n^+), \quad (7)$$

where

$$\widetilde{B}_n(p_1,\dots,p_n) = \frac{1}{(2\pi)^{2n}} \int \cdots \int \left(\prod_{j=1}^{n} d^4x_j\right) \exp\left[i\sum_{j=1}^{n} p_{jx}j\right] B_n(x_1,\dots,x_n). \tag{8}$$

By $S(\overline{\Omega}_n^+)$ is denoted the space of test functions
$\tilde{f}(p_1,\dots,p_n)$ with
$\operatorname{supp}\tilde{f}(p_1,\dots,p_n) \in \overline{\Omega}_n^+$, where

$$\overline{\Omega}_n^+(p_1,\dots,p_n) = \bigotimes_{j=1}^{n} \overline{\Omega}_\mu^+(p_j).$$

The existence of the generalized functions $M_n(x_1,\dots,x_n)$, $n = 1,2,\dots$, defined according to (7), follows from the existence of the generalized functions
$\widehat{B}_n(x_1,\dots,x_n) \in S_+^{\prime\mathrm{KT}}(R^{4n})$ and the definition

$$M_n(\tilde{f}_1,\dots,\tilde{f}_n) = \int \cdots \int \left(\prod_{j=1}^{n} d^4p_j\, \tilde{f}_j(p_j)\right) M_n(p_1,\dots,p_n) =$$

$$= (2\pi)^{-n/2}\widehat{B}_n(f_1,\dots,f_n) \tag{9}$$

for arbitrary $\tilde{f}_j(p_j) \in S(\overline{\Omega}_\mu^+)$, where
$f_j(x_j) \in S_+^{\mathrm{KT}}(R^4)$, $j = 1,2,\dots,n$, are determined according to (5).

2°. **Symmetry property.**

$$B_n(F) = 0, \tag{10}$$

where

$$F(x_1,\dots,x_n) = f(x_1,\dots,x_n) - f(x_{\alpha_1},\dots,x_{\alpha_n}) \tag{11}$$

for any $f(x_1,\dots,x_n) \in S(R^{4n})$.

3°. **Poincaré invariance**

$$B_n(f) = B_n(f_{(a,\Lambda)}), \tag{12}$$

where

$$f_{(a,\Lambda)}(x_1, \dots, x_n) = f(\Lambda^{-1}(x_1 - a), \dots, \Lambda^{-1}(x_n - a)) \tag{13}$$

for any $f(x_1, \dots, x_n) \in S(R^{4n})$ and $(a, \Lambda) \in P_+^\uparrow$, where a is a 4-translation vector, and Λ is an arbitrary element of the proper Lorentz group $L_+^\uparrow$.

B. **Nonlinear properties.**

4°. **Generalized unitarity relations**

$$i \sum_{s=0}^{n} P\left(\frac{f_1, \dots, f_s}{f_{s+1}, \dots, f_n}\right) \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{a_1 \dots a_l} B_{s+l}(f_1, \dots, f_s, h^*_{a_1}, \dots, h^*_{a_l}) \times$$

$$\times B^*_{n-s+l}(f_{s+1}, \dots, f_n, h_{a_1}, \dots, h_{a_l}) = \delta_{n0}, \quad n = 1, 2, \dots, \tag{14}$$

$$B_0 = B_0^* = 1$$

for arbitrary $f_j(x_j) \in S(R^4)$, $j = 1, \dots, n$, $\{h_\alpha(z)\}$ is a complete set of normalized functions of the space $S_+^{\mathrm{kr}}(R^{4n})$, representable in the form

$$h_{a_j}(z_j) = \frac{1}{(2\pi)^{3/2}} \int d^4 p_j \, \theta(p_j^0) \delta(p_j^2 - \mu^2) e^{ip_j z_j} \tilde{h}_{a_j}(p_j), \tag{15}$$

where $\tilde{h}_{a_j}(p_j) \in S(\bar{\Omega}_\mu^+)$, and $(h_\alpha, h_\beta) < \infty$,

$$(h_\alpha, h_\beta) = i \int d^3x \, f_\alpha(x) \overset{\leftrightarrow}{\partial}_0 f_\beta(x). \tag{16}$$

The Lorentz-invariant scalar product is

$$\sum_a h_a(x) h_a(x') = -i D^{(-)}(x' - x). \tag{17}$$

By $P\left(\frac{f_1, \dots, f_s}{f_{s+1}, \dots, f_n}\right)$ is denoted the symmetrization operator, meaning $n!/s!(n-s)!$ partitions of the set of n functions into two groups of s and $(n-s)$ functions in each.

5°. **Generalized causality relations**

$$(-i)^n B_{n+1}(f_0, \dots, f_n) + (i)^n \sum_{s=0}^{n-1} P\left(\frac{f_1, \dots, f_s}{f_{s+1}, \dots, f_n}\right) \times$$

$$\times \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{a_1 \dots a_l} B_{s+l+1}(f_0, \dots, f_s, h^*_{a_1}, \dots, h^*_{a_l}) \times$$

$$\times B^*_{n-s+l}(h_{a_1}, \dots, h_{a_l}, f_{s+1}, \dots, f_n) = 0, \tag{18}$$

$$B_0 = B^*_0 = 1$$

for arbitrary functions $f_j(x_j) \in S(R^4)$ with causally independent supports, i.e. satisfying the condition

$$f_0(x_0) f_j(x_j) = 0 \tag{19}$$

when

$$(x^0_0 - x^0_j) \leq 0 \quad \text{and} \quad (x_0 - x_j)^2 \geq 0, \quad j = 1, 2, \dots, n, \tag{20}$$

for at least one function $f_j(x_j) \in S(R^4)$, $j = 1, 2, \dots, n$, and for arbitrary functions $h_1(z_1), \dots, h_l(z) \in S(\bar{\Omega}^+_\mu)$, defined according to (15)–(17).

Restoration theorem. From an infinite set of numerical generalized B-functions satisfying conditions 1°–5°, one can construct the S-matrix according to

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \dots \int \left(\prod_{j=1}^{n} d^4 x_j \right) B_n(x_1, \dots, x_n) : \varphi_{\text{out}}(x_1) \dots \varphi_{\text{out}}(x_n) :, \tag{21}$$

which will:

a) commute with the unitary representation of the proper Poincaré group $P^\uparrow_+$, i.e.

$$[U(a, \Lambda), S] = 0, \tag{22}$$

where a is a vector of 4-translations, and Λ is an arbitrary proper transformation of the Lorentz group $L^\uparrow_+$,

b) unitary, i.e.

$$SS^+ = S^+ S = I; \tag{23}$$

c) satisfy the causality condition in the Bogoliubov form.

The generalized B-functions defined according to (3), and their linear and nonlinear properties 1°–5°, may henceforth be adopted as the basic mathematical conditions defining the axiomatic S-matrix BMP approach in quantum field theory, free of the original BMP formalism.

Now the elements of the S-matrix can be represented through generalized B-functions as follows:

$$S_{mn}(\mathbf{p}_1, \dots, \mathbf{p}_m, -\mathbf{q}_1, \dots, -\mathbf{q}_n) =$$
$$= \frac{(-i)^{m+n}}{(2\pi)^{\frac{3}{2}(m+n)}} \int \cdots \int \frac{\left(\prod_{r=1}^{n} d^4 x_r\right)\left(\prod_{s=1}^{m} d^4 y_s\right)}{\prod_{r=1}^{n}(2q_r^0)^{1/2} \prod_{s=1}^{m}(2p_s^0)^{1/2}} \times$$
$$\times \exp\left[-i\left(\sum_{r=1}^{n} q_{rx}r - \sum_{s=1}^{m} p_{sy}s\right)\right] B_{m+n}(x_1, \dots, x_n, y_1, \dots, y_m),$$
$$m, n = 0, 1, 2, \dots, \qquad (24)$$

where

$$q_r^0 = \sqrt{\mathbf{q}_r^2 + \mu^2}, \quad r = 1, 2, \dots, n, \qquad p_s^0 = \sqrt{\mathbf{p}_s^2 + \mu^2}, \quad s = 1, 2, \dots, m,$$
$$\sum_{r=1}^{n} q_r = \sum_{s=1}^{m} p_s$$

for all noncoincident momenta $p_1, \dots, p_m$ and $q_1, \dots, q_n$.

From the functional structure of the generalized B-functions it follows that

$$S_{mn}(p_1, \dots, p_m, -q_1, \dots, -q_n) \in S'(\mathbb{R}^{3(m+n)}).$$

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REFERENCES

1. B. V. Medvedev, *ZhETF*, **40**, 926 (1961).
2. B. V. Medvedev, M. K. Polivanov, International Winter School of Theoretical Physics at the Joint Institute for Nuclear Research, 1, 1964.
3. A. D. Sukhanov, *ZhETF*, **51**, 1195 (1966).

4. N. N. Bogoliubov, B. V. Medvedev, M. K. Polivanov, *Problems of the Theory of Dispersion Relations*, Moscow, 1958.

5. L. Sh. Khodzhaev, Preprint R5-3179, Joint Institute for Nuclear Research, Dubna, 1967.

6. L. Sh. Khodzhaev, *DAN*, **182**, No. 5 (1968).

Note: Figure translations are in progress. See original paper for figures.