ON SOME CLASSES OF TOPOLOGICAL SPACES AND THEIR BICOMPACT EXTENSIONS

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 513.831

MATHEMATICS

V. ZAĬTSEV

ON SOME CLASSES OF TOPOLOGICAL SPACES AND THEIR BICOMPACT EXTENSIONS

(Presented by Academician P. S. Aleksandrov on 23 V 1967)

In note (1) I called a T_λ**-space** any semiregular space X satisfying the following condition: whatever the point $x \in X$ and its neighborhood O_x, one can find canonical closed sets $A_1, \ldots, A_n$ such that

$$x \in A_1 \cap \cdots \cap A_n \subseteq O_x.$$

Here, as is known, a closed set is called **canonical** if it is the closure of its open kernel, and an open set—if it is the open kernel of its closure. By **semiregular** or T_ξ-spaces one means those T_1-spaces in which the canonical open sets form an open base (i.e., the canonical closed sets form a closed base). Let us further recall that a net of the space X (in the sense of A. V. Arhangel' skiĭ) is any such system $\mathfrak{M}$ of sets $M \subseteq X$ that, whatever the point $x \in X$ and its neighborhood O_x, there exists a set $M \in \mathfrak{M}$ satisfying the condition

$$x \in M \subseteq O_x.$$

Finally, let us call a π**-set** of the space X any set that is the intersection of a finite number of canonical closed sets. Thus, T_λ-spaces are precisely those T_ξ-spaces in which the π-sets form a net.

In the same note (1), in particular, for every T_1-space X there is considered a bicompact space $\omega_\kappa X$, which I called the Urysohn-Ponomarev space: the points of this space are κ-ends, i.e. maximal centered systems $\xi = \{A_\alpha\}$ of canonical closed sets; the topology in $\omega_\kappa X$ is the Urysohn topology. I proved in (1) that, in particular, for every T_λ-space X the space $\omega_\kappa X$ is a bicompact T_ξ-space which is the limit of a (maximal finite) spectrum S_X of the space X, and that for a T_λ-space X the space $\omega_\kappa X$ is an extension of the space X. I can now supplement these results with the following new ones.

First of all, it follows from Lemma 3 of note (1) that the class of bicompact T_λ-spaces coincides with the class of bicompact T_ξ-spaces. Therefore, for a T_λ-space X, the space $\omega_\kappa X$ will be not only a T_ξ- but also a T_λ-space. Hence it follows that Theorem 5 of note ([1]) may be formulated in the following more complete form:

Theorem 1. *All T_λ-spaces and only T_λ-spaces have bicompact T_λ-extensions.*

Let us call a space X **quasinormal** if it is regular and if, in it, any two disjoint π-sets have disjoint neighborhoods that are canonical open sets. It is easy to show* that every quasinormal space is completely regular; at the same time every normal space is, obviously, quasinormal. Thus the class of quasinormal spaces is situated strictly between the class of completely regular spaces and the class of normal spaces, coinciding with neither of these classes.

* On the basis of Theorem 1 of work ([3]).

Theorem 2. In order that the space $\omega_* X$ be Hausdorff (and consequently, being bicompact, normal), it is necessary and sufficient that X be quasinormal.

Theorem 3. For every completely regular space X, the space $\omega_* X$ is continuously mapped, with the points of X fixed, onto any Hausdorff bicompact extension of the space X.

From the last two results it follows that

Corollary. For quasinormal spaces X, and only for them, we have $\omega_* X = \beta X$ (where βX, as always, is the Čech–Stone extension of the space X).

These results give the definitive solution of a problem posed long ago by P. S. Aleksandrov:

Theorem 4. For quasinormal spaces, and only for them, the limit of the maximal finite spectrum S_X is βX.

It is easy to see that every extremally disconnected space is quasinormal; therefore, from what has been proved there follows the following assertion, also contained in the works of V. I. Ponomarev (for example, in ([2])).

The limit of the spectrum S_X of every extremally disconnected space X is the space βX.

Remark. The first assertion of Theorem 6 of my note can be somewhat strengthened, namely:

Among all bicompact T_λ-extensions of a given T_λ-space,* the extension $\omega_* X$ is the unique maximal one in the sense that the spectrum of any T_λ-extension $\overline{X}$ is a refinement of the spectrum $\omega_* X$ (which coincides with the spectrum of X itself). An inaccuracy crept into the second assertion of Theorem 6 of note ([1]); after correction, this assertion becomes Theorem 3 of the present note.

Mechanics and Mathematics Faculty
Moscow State University
named after M. V. Lomonosov

Received
11 V 1967

CITED LITERATURE

[1] V. Zaĭtsev, DAN, **171**, No. 3, 521 (1966).
[2] V. Ponomarev, DAN, **149**, No. 1, 26 (1963).
[3] V. Zaĭtsev, Vestn. Mosk. Univ., No. 3, 48 (1967).

* And not only of a regular space.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.