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Abstract

Full Text

MATHEMATICS

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INVESTIGATION OF A NONLINEAR SINGULAR INTEGRAL EQUATION WITH A SPECIAL CAUCHY KERNEL IN THE CLASS $H^k_{\alpha-\delta,\delta}(a,b)$

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In the present note we investigate the equation

$$U(x) = \lambda(x-a)^\alpha \int_a^b \frac{f[x,s,U(s)]}{(s-a)^\alpha(s-x)}\,ds \tag{1}$$

in the space $H^k_{\alpha-\delta,\delta}(a,b)$, where $0<\alpha<1$.

Definition. $U(x) \in H^k_{\alpha-\delta,\delta}(a,b)$ if the function $U(x)$ is defined for $a \le x < b$ and satisfies the conditions:

$$|U(x)| \le K(x-a)^\delta/(b-x)^{\alpha-\delta},$$

$$|U(x+\Delta x)-U(x)| \le K|\Delta x|^\delta/(b-x-\Delta x)^\alpha,$$

where $0<\delta<\alpha<1,\ 0<|\Delta x| \le \min\{|x-a|/4,\ |b-x|/4\},\ K>0.$

For the exposition of the main result we state, without proof, several lemmas.

Lemma 1. If the function $\varphi(x,s)$ is defined for $a \le x \le b,\ a \le s < b$ and satisfies the conditions

$$|\varphi(x,s)| \le L(s-a)^\delta/(b-s)^{\alpha-\delta},$$

$$|\varphi(x+\Delta x, s+\Delta s) - U(x,s)| \le L(|\Delta x|^{\delta_1}+|\Delta s|^\delta)/(b-s-\Delta s)^\alpha,$$

then the function

$$W(x,s) = (s-a)^{-\alpha}\varphi(x,s)$$

for all $a \leq x \leq b,\ a < s < b$ and $0 < |\Delta s| \leq \min\{|s-a|/4,\ |b-s|/4\}$ satisfies the conditions:

$$|W(x,s)| \leq LL_1/(s-a)^{\alpha-\delta}(b-s)^{\alpha-\delta},$$

$$|W(x+\Delta x, s+\Delta s) - W(x,s)| \leq$$

$$\leq LL_1(|\Delta x|^{\delta_1} + |\Delta s|^{\delta})/(s-a)^{\alpha}(b-s-\Delta s)^{\alpha},$$

where $0 < \delta < \alpha < 1,\ \delta < \delta_1,\ L_1$ is an absolute constant.

Lemma 2. If the function $\varphi(x,s)$ is defined for $a \leq x \leq b,\ a < s < b$ and satisfies the conditions:

$$|\varphi(x_1,s)| \leq B/(s-a)^{\alpha-\delta}(b-s)^{\alpha-\delta},$$

$$|\varphi(x+\Delta x, s+\Delta s) - \varphi(x,s)| \leq$$

$$\leq B(|\Delta x|^{\delta_1} + |\Delta s|^{\delta})/(s-a)^{\alpha}(b-s-\Delta s)^{\alpha},$$

then the function

$$V(x,s) = (s-a)^{\alpha}\varphi(x,s)$$

for all $a \leq x \leq b,\ a \leq s < b$ and $0 < |\Delta s| \leq \min\{|s-a|/4,\ |b-s|/4\}$

satisfies the conditions

$$|V(x,s)| \leq BB_1(s-a)^{\delta}/(b-s)^{\alpha-\delta},$$
$$|V(x+\Delta x, s+\Delta s) - V(x,s)| \leq BB_1(|\Delta x|^{\delta_1} + |\Delta s|^{\delta})/(b-s-\Delta s)^{\alpha},$$

where $B_1 = \text{const}$.

Lemma 3. If the function $f(x,s,u)$ is defined for $a \leq x,\ s < b, -\infty < u < +\infty$, and satisfies the conditions

$$|f(x+\Delta x, s+\Delta s, u+\Delta u) - f(x,s,u)| \leq M_2(|\Delta x|^{\delta_1} + |\Delta s|^{\delta})/(b-s-\Delta s)^{\alpha} + M_2|\Delta u| \quad (2)$$

and the function

$$\omega(x) = (x-a)^\alpha \int_a^b \frac{f(x,s,0)}{(s-a)^\alpha (s-x)}\, ds$$

belongs to $H^{k_1}_{\alpha-\delta,\delta}(a,b)$, then for all $a \le x < b$ and

$$|\lambda| < K/(K_1 + M_3 L_1 B_1 L_2)$$

the operator

$$Au = \lambda (x-a)^\alpha \int_a^b \frac{f[x,s,U(s)]}{(s-a)^\alpha (s-x)}\, ds \tag{3}$$

maps the space $H^k_{\alpha-\delta,\delta}(a,b)$ into itself, where

$$M_3 = \max\{KM_2,\ 2M_1 + KM_2\};$$

L_2 is a constant independent of M_3, L_1, B_1.

Lemma 4. If the function $f(x,s,u)$ is defined for $a \le x,\ s < b$, $-\infty < u < +\infty$, and satisfies condition (2), and $U(x) \in H^k_{\alpha-\delta,\delta}(a,b)$, then for

$$\delta_1 \ge \delta + 2\alpha, \qquad p > \max\{1/(1-\alpha-\delta),\ 1/(\delta_1 - \delta - 2\alpha)\}$$

the operator (3) is continuous in the sense of the metric $L_p(\rho_1)$, where

$$\rho_1(s) = (b-s)^{(\alpha+\delta)p}, \qquad 0 < \alpha + \delta < 1.$$

We note that the set $H^k_{\alpha-\delta,\delta}(a,b)$ is a closed, convex, compact subset of the space $L_p(\rho_1)$ for any $p > 1$. Consequently, on the basis of the generalized Schauder principle there exists a fixed point, i.e., the following holds.

Theorem 1. If the function $f(x,s,u)$ satisfies the conditions of Lemmas 3 and 4, then there exists a number

$$\bar{\lambda} = K/(K_1 + M_3 L_1 B_1 L_2),$$

such that for $|\lambda| < \bar{\lambda}$ equation (1) has at least one solution

$$U(x) \in H^k_{\alpha-\delta,\delta}(a,b).$$

One can also show the uniqueness of the solution of equation (1).

If the function $f(x,s,u)$ satisfies the conditions of Lemma 3 and, in addition, the function

$$K(x,s,u) = f(x,s,u) - f(s,s,u)$$

satisfies the condition

$$|K(x,s,u) - K(x,s,v)| \le C|x-s|^{\alpha_1}|u-v| \tag{4}$$

for all $a \le x, s < b$, $-\infty < u < \infty$, $0 < 1 - \alpha_1 + \alpha + \delta < 1$, then the operator (3) acts from $L_p(\rho_1)$ into $L_p(\rho_1)$ and satisfies the Lipschitz condition

$$\|Au - Av\|_{L_p(\rho_1)} \le |\lambda|(RC + M_2F)\|u-v\|_{L_p(\rho_1)},$$

where

$$1 < q < q_0; \qquad 1/p + 1/q = 1; \qquad 0 < (1 - \alpha_1 + \alpha + \delta)q_0 < 1;$$

$$R = \left\{ \int_a^b s(x) \left[\int_a^b s^{-q/p}(s)|x-s|^{q(\alpha_1-1)}\, ds \right]^{p/q} dx \right\}^{1/p}; \qquad \rho(x) = (b—x)^{(\alpha+\delta)p}(x—a)^{\alpha p}.$$

F is a constant independent of $U(s)$.

Since $H^k_{\alpha-\delta,\delta}(a,b)$ is complete in the sense of the metric $L_p(\rho_1)$, the following is true.

Theorem 2. If the function $f(x,s,u)$ satisfies the conditions of Theorem 1 and condition (4), then, for

$$|\lambda| < \bar{\lambda}_0 = \min\{K/(K_1 + M_3L_1B_1L_2),\ 1/(RC + M_2F)\},$$

equation (1) has a unique solution $U^*(x) \in H^k_{\alpha-\delta,\delta}(a,b)$. It can be found by the method of successive Picard approximations. The successive approximations will converge in the sense of the metric of the space $L_p(\rho_1)$; moreover, if

$$U_n(x) = \lambda(x-a)^\alpha \int_a^b \frac{f[x,s,U_{n-1}(s)]}{(s-a)^\alpha(s-x)}\, ds, \qquad U_0(x) \in H^k_{\alpha-\delta,\delta}(a,b),$$

then

$$\rho_{L_p(\rho_1)}(U_n, U_0^*) < \frac{\lambda_0^n}{1-\lambda_0}\gamma,$$

where γ is a constant independent of n, $0 < \lambda_0 = |\lambda|(RC + M_2F) < 1$.

Moreover, if (2) is taken into account, then the successive approximations also converge in the sense of the metric $\rho_{\delta'}$, and the estimate

$$\rho_{\delta'}(U_n, U_0^*) \leq l(K_2)^{1/(1+\delta p)} \frac{\lambda_0^{n\delta p/(1+\delta p)}}{(1-\lambda_0)^{\delta p/(1+\delta p)}} \gamma^{\delta p/(1+\delta p)} +$$

$$+(2l)^{(\delta-\delta')/\delta} (K_2)^{(1+\delta' p)/(1+\delta p)} \frac{\lambda_0^{(\delta-\delta')pn/(1+\delta p)}}{(1-\lambda_0)^{(\delta-\delta')p/(1+\delta p)}} \gamma^{(\delta-\delta')p/(1+\delta p)}$$

holds, where $K_2 = Kl_1$; $l_1 = \text{const}$; l is a constant independent of K_2.

If $U(x, \lambda_1)$ and $U(x, \lambda_2)$ are solutions of equations (1) corresponding to the parameters λ_1 and λ_2 ($|\lambda_1| < \lambda_0,\ |\lambda_2| < \lambda_0$), and the function $f(x, s, u)$ satisfies the conditions of Theorem 2, then it can be proved that the solution depends continuously on λ.

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REFERENCES

1. A. I. Guseinov, *Izv. AN SSSR, Ser. Mat.*, **12**, 193 (1948).

2. A. I. Guseinov, Kh. Sh. Mukhtarov, *DAN*, **168**, No. 5 (1966).

3. B. V. Khvedelidze, *Tr. Tbilissk. Mat. Inst.*, **23** (1956).

4. A. A. Babaev, *Uch. Zap. Azerb. Gos. Univ. im. S. M. Kirova*, No. 2 (1960).

5. L. V. Kantorovich, G. P. Akilov, *Functional Analysis in Normed Spaces*, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.