ON THE CAUCHY PROBLEM FOR A SECOND-ORDER HYPERBOLIC EQUATION DEGENERATING ON THE INITIAL HYPERPLANE

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Abstract

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A. B. NERSESIAN

ON THE CAUCHY PROBLEM FOR A SECOND-ORDER HYPERBOLIC EQUATION DEGENERATING ON THE INITIAL HYPERPLANE

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The Cauchy problem for a second-order hyperbolic equation with data on a surface of degeneracy had until recently been studied only in the plane case (see the bibliography in ([1])). The results of ([1]) make it possible to form a fairly complete picture of the conditions for the correctness of this problem in the case when the characteristics of a second-order equation (or system) intersect the line of parabolicity at a nonzero angle.

In the multidimensional case these questions have been covered with sufficient completeness only in the work of O. A. Oleinik ([2]), where the Cauchy problem and the boundary-value problem for the equation

$$-u_{tt} + \left(a^{ij}(x,t)u_{x_i}\right)_{x_j} + b^i(t,x)u_{x_i} + c(t,x)u_t + d(t,x)u = f(t,x) \quad (t > 0) \quad (1)$$

with initial data at $t = 0$ are considered, the form

$$a^{ij}(t,x)\xi_i\xi_j \geqslant 0 \quad (i,j = 1,2,\dots,n) \quad (2)$$

being allowed to degenerate both on the boundary and inside the corresponding domain of definition of the solution.

The main restriction imposed in ([2]) on the coefficients of equation (1) is the following:

$$Aa^{ij}\xi_i\xi_j + \dot{a}^{ij}\xi_i\xi_j - \alpha(b^i\xi_i)^2 \geqslant 0, \quad (3)$$

where $A > 0$ and $\alpha > 0$ are certain constants.

This condition does not allow the form (2) to degenerate on sets of arbitrary form. For example, the possibility of degeneration only at a single interior point of the domain is excluded (if, for fixed x and ξ, $a^{ij}(t_0,x)\xi_i\xi_j = 0$ for $t_0 > 0$, then from condition (3) it follows that $a^{ij}\xi_i\xi_j \equiv 0$ for $0 \leqslant t \leqslant t_0$).

It remains unclear whether the Cauchy problem for equation (1) will be correct (at least in a generalized sense, and even in the plane case) if the form (2) is allowed to degenerate on more or less arbitrary sets inside the domain of definition of the solution, while certain other restrictions are imposed on the coefficients b^i (and even if one assumes that $b^i \equiv 0$).

On the other hand, comparison with the corresponding conditions of ([1]) shows that, in the case of degeneration only on the hyperplane $t = 0$, condition (3) is too restrictive. Other results relating to this case (see ([3])) are based on still more restrictive assumptions.

Below, for an equation with coefficients depending only on time, the results of ([1]) are extended to the multidimensional case.

1°. Consider the equation

$$-u_{tt}+a^i(t)u_{tx_i}+a^{ij}(t)u_{x_ix_j}+b^i(t)u_{x_i}+c(t)u_t+d(t)u = f(t,x) \quad (i,j=1,2,\dots,n). \tag{4}$$

For $t > 0$ the equation is hyperbolic, and degeneracy is allowed only on the hyperplane $t = 0$, where the homogeneous initial data of the Cauchy problem are prescribed:

$$u(+0,x) = 0, \qquad u_t(+0,x) = 0, \qquad x \in \Omega. \tag{5}$$

By applying the method of expansion into plane waves (see, for example, ($4'$)), this problem is formally reduced to the corresponding problem in the plane case. Indeed, introduce the Radon transform of the unknown function

$$V^\xi(t,p) = \int_{x\cdot\xi=p} u(t,x)\,dS_x \qquad (\xi = (\xi_1,\dots,\xi_n)). \tag{6}$$

Let D denote the natural domain of definition of the solution of equation (4) corresponding to the "base" Ω. By hyperbolicity, one may assume that outside D, $f \equiv 0$ and $u \equiv 0$, while on the boundary of D the solution u, together with its derivatives up to second order with respect to the spatial variables, vanishes. It is also obvious that Ω may be assumed to be a convex domain (and even a ball) on the hyperplane $t = 0$.

Thus equation (4) reduces to the following:

$$-V^\xi_{tt} + (a^i\xi_i)V^\xi_{tp} + (a^{ij}\xi_i\xi_j)V^\xi_{pp} + (b^i\xi_i)V^\xi_p + cV^\xi_t + dV^\xi = \int_{x\cdot\xi=p} f(t,x)\,dS_x. \tag{7}$$

This is a hyperbolic equation in the half-plane $t > 0$, whose coefficients and right-hand side depend on the parameter ξ, and it degenerates on the line $t = 0$. It is obvious that the solution V^ξ satisfies homogeneous Cauchy conditions on some segment of the axis $t = 0$. If we solve this problem, then the solution u of problem (4)–(5) can formally be recovered by the formula (4′)

$$2(2\pi i)^{n-1} u(t,x) = (\Delta_x)^{(n-1)/2} \int_{\omega_\xi} V^\xi(t, \xi \cdot x)\, d\sigma_\xi, \tag{8}$$

where ω_ξ is the spherical surface of unit radius in the n-dimensional ξ-space; σ_ξ is the element of its surface, and Δ_x is the Laplace operator with respect to $x = (x_1, \dots, x_n)$. Here and below, n is an odd number; however, this restriction is inessential, since in the case of even n the corresponding results are easily carried over by the method of descent.

2°. In order to make use of the calculations of the preceding subsection, we must require that the function V^ξ be at least $(n+1)/2$ times differentiable with respect to p and continuous with respect to ξ. It turns out that if the corresponding integral (10) of paper (1), which in the present case depends on the parameter ξ, converges uniformly in ξ, then the existence and uniqueness of the function V^ξ with the required properties are ensured and, moreover, the solution V^ξ is stable with respect to changes in f, uniformly in ξ ($\xi_i \xi_i = 1$).

To formulate the corresponding result, introduce the notation

$$(\Lambda^\xi)^2 = (a^i \xi_i)^2 + a^{ij} \xi_i \xi_j;$$

$$\begin{gathered} \alpha_1^\xi = \Delta_t^\xi + (b^i - c a^i + a_t^i)\xi_i; \\ \alpha_2^\xi = \alpha_1^\xi - 2\Delta_t^\xi; \qquad 2\Delta^\xi \alpha^\xi = |\alpha_1^\xi| + |\alpha_2^\xi|; \end{gathered} \tag{9}$$

$$f_p^k(t,\xi) = \int_0^t |\alpha_k^\xi(t_1)|(t - t_1) \dots \int_0^{t_p} |\alpha_k^\xi(t_{p+1})|(t_p - t_{p+1})\, dt_{p+1} \dots dt_1$$
$$(k = 1, 2;\ p \geqslant 0). \tag{10}$$

Theorem. Let the function $f(t,x)$ be continuous in $\overline{\mathcal{D}}$ together with its derivatives with respect to x up to order $2p + (n+5)/2$; let the functions $a^i(t)$ and $a^{ij}(t)$ be continuously differentiable for $t \geqslant 0$, and let the functions $b^i(t)$, $c(t)$, and $td(t)$ be summable in any neighborhood of zero ($t \geqslant 0$). If, in at least one of the two cases ($k = 1, 2$),

$$\int_0^h t\Delta^\xi(t)\, d\left\{ f_p^k(t,\xi) \exp\left[\int_t^h \alpha^\xi(\tau)\, d\tau\right]\right\} < +\infty \qquad (h > 0), \tag{11}$$

and the convergence is uniform with respect to ξ ($\xi_i\xi_i = 1$), then problem (4)–(5) has a unique solution possessing in $\mathcal{D}$ continuous derivatives

$$\frac{\partial^2}{\partial t^2} D^\alpha u \quad (\alpha = (\alpha_1, \dots, \alpha_n),\ |\alpha| = p).$$

This solution is stable in the following sense: let solutions u^i correspond to functions f^i ($i = 1, 2$). Then for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that from the estimate

$$\sum_{|\alpha|<(n+5)/2+2p} \max_{\mathcal{D}} \left| D^\alpha (f^1 - f^2) \right| < \delta \tag{12}$$

it follows that

$$\sum_{i=0}^{2} \sum_{|\alpha| \leqslant p+2-i} \max_{\mathcal{D}} \left| \frac{\partial^i}{\partial t^i} D^\alpha (u^1 - u^2) \right| < \varepsilon. \tag{12'}$$

3°. Condition (11) is certainly satisfied for some $p \geqslant 0$ if

$$\int_t^h \frac{|\Delta_t^\xi(t)|}{\Delta^\xi(t)}\, dt \leqslant \text{const} \cdot \log \frac{1}{\Delta^\xi(t)} \qquad (0 < t < h); \tag{13}$$

$$\int_0^t |\Delta_t^\xi(t)|\, dt \leqslant \text{const} \cdot \Delta^\xi(t) \qquad (0 < t < h); \tag{13'}$$

$$(b^k\xi_k)^2 (a^{ij}\xi_i\xi_j) \leqslant \text{const}\, (a_t^{ij}\xi_i\xi_j)^2 \qquad (0 < t < h), \tag{14}$$

where the constants on the right depend only on h.

Let, for example,

$$a^{ij}(t) = \delta^{ij} t^{2\beta} (2 + \cos\log t^{-\alpha})^2 \qquad (0 < 2\beta < \alpha), \tag{15}$$

where δ^{ij} is the Kronecker symbol. In the present case it is not hard to verify that conditions (13)–(13′) are satisfied, while condition (3) is violated even when $b^i \equiv 0$. From (14) it follows that the theorem of item 2° is valid, for an appropriate choice of p, if as $t \to 0$

$$b^i(t) = O(t^{\beta-1})$$

(i.e., for $\beta < 1$ the functions $b^i(t)$ may even increase without bound).

On the other hand, even when $a_t^{ij}\xi_i\xi_j \geqslant 0$ (i.e., when $\Delta_t^\xi \geqslant 0$), when the estimates (13)–(13′) are trivial, condition (14) means that as $t \to 0$ the function

$(b^i\xi_i)^2$ is of order $(\Delta_t^\xi)^2$, whereas from condition (3) it follows that this function must be of order $\Delta^\xi\Delta_t^\xi$. Thus, condition (3) is the more restrictive than condition (11), the faster $\Delta^\xi(t)$ tends to zero as $t \to 0$.

We also note the particular case when $a_t^{ij}\xi_i\xi_j \geq 0$ (i.e., $\Delta_t^\xi \geq 0$) and

$$a^{ij}\xi_i\xi_j \geq a^2(t)\xi_i\xi_i, \qquad |b^i| \leq b_0^i a_t(t) \quad (t > 0), \tag{16}$$

where $a(t) > 0$, $a'(t) \geq 0$ for $t > 0$, $a(0) = 0$, and $b_0^i = \text{const}$. Then condition (11) is satisfied if

$$\int_0^h t^{p+2}[a(t)]^{1+p-q}\,da(t) < +\infty \qquad (q^2 = b_0^i b_0^i,\ h > 0), \tag{17}$$

whence follows the well-posedness of the corresponding Cauchy problem if $p \geq q - 1$.

4°. By the same method, the results of item 5° of the work [1] carry over to the following Cauchy problem for a model second-order system:

$$u_t^k + a_k^{ij}(t)u_{x_j}^i = b_k^i(t)u^i + f^k(t,x) \qquad (k = 1,2;\ t > 0) \tag{18}$$

$$u^k(+0,x) = 0, \qquad x \in \Omega, \tag{18'}$$

hyperbolic for $t > 0$ and degenerating at $t = 0$. Denote

$$(\Delta^\xi)^2 = [(a_1^{1i} - a_2^{2i})\xi_i]^2 + 4a_1^{2i}a_2^{1j}\xi_i\xi_j;$$

$$\omega^\xi(a_2^{1i}\xi_i) = \Delta^\xi, \qquad 2(a_2^{1i}\xi_i)c^\xi = (a_1^{1i} - a_2^{2i})\xi_i + \Delta^\xi;$$

$$\alpha_1^\xi = b_1^1 + c_t^\xi + c^\xi(b_1^1 - c^\xi b_2^1 - b_2^2); \qquad \alpha_2^\xi = \alpha_1^\xi + \omega_t^\xi; \tag{19}$$

$$\omega^\xi\alpha^\xi = |\alpha_1^\xi| + |\alpha_2^\xi|; \qquad \beta^\xi = a_2^{1i}\xi_i;$$

$$f_p^i(t) = \int_0^t \beta^\xi(t_1)\int_0^{t_1} \alpha_i^\xi(t_2)\cdots\int_0^{t_{2p}} \beta^\xi(t_{2p+1})\int_0^{t_{2p+1}} \alpha_i^\xi(t_{2p+2})\,dt_{2p+2}\cdots dt_1$$

$$(i = 1, z;\ p \geq 0). \tag{20}$$

The assertions of Theorem item 2° remain valid also for the system (18)–(18′), if, for $\beta^\xi > 0$ ($t > 0$), the functions f^k possess continuous derivatives with respect to x in $\overline{D}$ up to order $(n+5)/2+2p$ ($p \geq 0$), the functions $a_k^{ij}(t)$ are continuously differentiable for $t \geq 0$, the functions $b_k^i(t)$ are summable in a neighborhood of zero, and, at least in one of the two cases ($k = 1, 2$), uniformly with respect to ξ,

$$\int_0^h t\,\omega^\xi(t)\,d\left\{f_p^k(t)\exp\left[\int_t^h \alpha^\xi(\tau)\,d\tau\right]\right\} < +\infty \qquad (h > 0). \tag{21}$$

5°. The requirement that the coefficients of equations (4) and (18) be independent of the spatial variables is dictated in the present case only by the application of the Radon transform. Apparently, as in the planar case, the results remain valid even without this restriction (in any case, this seems quite possible if only the coefficients at the highest derivatives do not depend on x).

Institute of Mathematics and Mechanics
Academy of Sciences of the Armenian SSR

Yerevan State University

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Note: Figure translations are in progress. See original paper for figures.