MULTIVALUED MAPPINGS AND BOREL SETS

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Let X be a topological space. Denote by $F(X)$ the space of all nonempty closed subsets of the space X in the Vietoris topology. In the space $F(X)$ let us single out the subspace

$$C(X) = \{L \in F(X) \mid L \text{ is bicompact}\}.$$

A single-valued mapping $\theta : Y \to F(X)$ from the space Y into the space $F(X)$ is called a multivalued mapping. With multivalued mappings there is associated the problem of sections (see ([4],[5])): for which multivalued mappings $\theta : Y \to F(X)$ does there exist a "good" single-valued mapping $f : Y \to X$ such that $fy \in \theta y$ for every point $y \in Y$. Recall that a mapping $\theta : Y \to F(X)$ is called upper (lower) semicontinuous if for every open (closed) set $A \subset X$ the set

$$\{y \in Y \mid \theta y \subset A\}$$

is open (closed) in the space Y. A mapping $\theta : Y \to F(X)$ is continuous if it is at the same time upper and lower semicontinuous.

§ 1. **On canonical representation.** Let $\theta : X \to F(Y)$ be a multivalued mapping. The set

$$\Gamma(X, Y, \theta) = \bigcup \{(\{x\} \times \theta x) \mid x \in X\} \subset X \times Y$$

is called the graph of the mapping θ. Denote by π_X and π_Y the natural projections of the set Γ onto X and onto Y, respectively. We have $\theta = \pi_Y \circ \pi_X^{-1}$. We shall say that the mapping θ is X-open (X-closed) if the mapping π_X is open (closed). If π_X is perfect, then θ is called X-perfect.

Proposition 1. The following properties are equivalent: a) the mapping $\theta : X \to F(Y)$ is X-perfect; b) the mapping θ is upper semicontinuous and $\theta x \in C(Y)$ for every point $x \in Y$.*

Proposition 2. *The following properties are equivalent: a) the mapping $\theta : X \to F(Y)$ is X-open; b) the mapping θ is lower semicontinuous.*

These propositions establish a connection between single-valued and multivalued mappings. Let us note that there exists an upper semicontinuous mapping $\theta : X \to F(Y)$ which is not X-closed.

§ 2. **Multivalued continuous mappings.**

Theorem 1. *Let the mapping $\theta : X \to F(Y)$ be continuous, where Y is a complete zero-dimensional metric space. Then there exists a single-valued continuous mapping $f : X \to Y$ such that $fx \in \theta x$ for every point $x \in X$.*

Corollary 1. *Let $\varphi : X \to Y$ be an open-and-closed single-valued continuous mapping of a complete zero-dimensional metric space X onto Y. Then there exists a closed subset $X_1 \subset X$ such that $\varphi X_1 = Y$ and $\varphi|X_1$ is a homeomorphism.*

Theorem 2. *Let the mapping $\theta : X \to F(Y)$ be continuous, where Y is a complete metric space. Then there exists a single-valued mapping $f : X \to Y$ such that $fx \in \theta x$ for every point $x \in X$, and $f^{-1}U$ is an F_σ-set whenever U is open in Y.*

Theorem 3. *Let $\theta : X \to F(Y)$ be a continuous mapping of a normal space X into a complete metric space Y. Then there exist mappings $\varphi : X \to C(Y)$ and $\psi : X \to C(Y)$ such that: a) $\varphi x \subset$*

$\subset \psi x \subset \theta x$ for every point $x \in X$; b) the mapping ψ is upper semicontinuous; c) the mapping φ is lower semicontinuous; d) if, moreover, $\dim X = 0$, then the mapping ψ is single-valued, i.e., φx and ψx are singletons.

In the paper (^{7}) E. Michael proved an analogous theorem for lower semicontinuous mappings under the assumption that the space X is paracompact. Let us note that the assertion converse to Michael' s theorem is true.

Theorem 4. *Let X be a topological T_1-space. If for every lower semicontinuous mapping $f : X \to F(Y)$, where Y is an arbitrary complete metric space, there exists an upper semicontinuous mapping $\theta : X \to C(Y)$ such that $\theta x \subset fx$ for every point $x \in X$, then the space X is paracompact.*

Proof. First, let us prove the normality of the space X. Let A and B be arbitrary closed and disjoint subsets of the space X. Let $Y = \{a, b\}$ be an ordinary two-point set (with the discrete topology). We construct a mapping $f : X \to F(Y)$, where

$$fx = \begin{cases} a, & \text{if } x \in A, \\ b, & \text{if } x \in B, \\ Y, & \text{if } x \in X \setminus (A \cup B). \end{cases}$$

The mapping f is lower semicontinuous, since $f^{-1}a = X \setminus B$ and $f^{-1}b = X \setminus A$. By assumption there exists an upper semicontinuous mapping $\theta : X \to C(Y)$

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* This proposition is a strengthening of a lemma of Yu. M. Smirnov and V. I. Ponomarev on the canonical representation of perfect mappings (see (^{8})).

such that $\theta x \subset fx$ for every point $x \in X$. Put $U = \{x \in X \mid \theta x \subset \{a\}\}$ and $V = \{x \in X \mid \theta x \subset \{b\}\}$. By construction $U \cap V = \emptyset$, $U \supset A$, and $V \supset B$. From the definition of upper semicontinuity it follows that the sets U and V are open in the space X. The normality of X is proved.

Now let $\omega = \{U_\alpha \mid \alpha \in Y\}$ be some open cover of the space X. On the set Y define a metric in the following way: $\rho(\alpha, \beta) = 1$ if $\alpha \neq \beta$, and $\rho(\alpha, \alpha) = 0$. Put $f : X \to F(Y)$, where $fx = \{\alpha \in Y \mid x \in U_\alpha\}$. The mapping f is lower semicontinuous, since $f^{-1}\alpha = U_\alpha$ for every $\alpha \in Y$. By assumption, there exists an upper semicontinuous mapping $\theta : X \to C(Y)$ such that $\theta x \subset fx$ for every point $x \in X$.

Put $\gamma = \{F_\alpha = \theta^{-1}\alpha = \{x \in X \mid \theta x \cap \{\alpha\} \neq \emptyset\} \mid \alpha \in Y\}$. The system γ covers the space X. Since $F_\alpha \subset U_\alpha$ for every $\alpha \in Y$, the cover γ is inscribed in the cover ω. Let $x_0 \in X$. Since the space Y is discrete, the set $\theta x_0 = \{\alpha_1(x), \dots, \alpha_{n(x)}\}$ is finite and open. Consequently, the set $Ox_0 = \{x \in X \mid \theta x \subset \theta x_0\}$ is also open in the space X and has nonempty intersection only with the sets F_α where $\alpha \in \theta x_0$. Since the set θx_0 is finite and $x_0 \in Ox_0$, the cover γ is locally finite. On the basis of a known theorem of E. Michael from (^{10}), the space X is paracompact.

Theorem 3 and Theorem 3.2 from (^{5}) allow us to obtain the following theorem.

Theorem 5. *Let $\theta : X \to F(Y)$ be a continuous mapping of a collectionwise normal space X into a Banach space Y. Then there exists a single-valued continuous mapping $f : X \to Y$ such that $fx \in \operatorname{conv}(\theta x)$ for every point $x \in X$.*

§ 3. **Lower semicontinuous mappings.** Let X be a topological space. Put

$$B_1(X) = \{A \subset X \mid A = L \setminus C;\ L, C \in F(X)\}$$

and

$$B(X) = \{K \subset X \mid K = \bigcup_{n=1}^{\infty} A_n,\ A_n \in B_1(X) \text{ and } n = 1, 2, \dots\}.$$

It is easy to see that, in the case of perfectly normal spaces, the system $B(X)$ coincides with the system of sets of type F_σ.

Theorem 6. *Let $\theta : X \to C(Y)$ be a lower semicontinuous mapping, where Y is a metrizable space. Then there exists a single-valued mapping $f : X \to Y$ such that $fx \in \theta x$ for every point $x \in X$, and $f^{-1}U \in B(X)$ for every open set U in Y.*

Theorem 7. For every regular space X the following conditions are equivalent: a) the space X is weakly paracompact; b) for every lower semicontinuous mapping $\theta : X \to F(Y)$, where Y is an arbitrary complete metric space, there exists a lower semicontinuous mapping $\psi : X \to C(Y)$ such that $\psi x \subset \theta x$ for every point $x \in X$.

From Theorems 6 and 7 we obtain:

Theorem 8. Let $\theta : X \to F(Y)$ be a lower semicontinuous mapping of a weakly paracompact regular space X into a complete metric space Y. Then there exists a single-valued mapping $f : X \to Y$ such that $fx \in \theta x$ for every point $x \in X$, and $f^{-1}U \in B(X)$ for every open set U in Y.

Theorem 7 allows us to definitively generalize a theorem of E. Michael from (7).

Theorem 9. Let $f : X \to Y$ be an inductively open single-valued mapping* of a metric space X onto a regular weakly paracompact space Y, with complete point-preimages (in the metric given on X). Then there exists a subspace $X_1 \subset X$ such that $fX_1 = Y$ and the mapping $f|X_1$ is open and bicompact.

Theorem 10. Let the mapping $\theta : X \to F(Y)$, where Y is a complete metric space, be lower semicontinuous. If the space X satisfies one of the following conditions: a) X is paracompact; b) it is perfectly normal and weakly paracompact; c) X is symmetrizable with the first axiom of countability; d) X has a σ-discrete net; e) X is perfectly normal and into every open cover one can inscribe a closed σ-discrete cover, then there exists a single-valued mapping $f : X \to Y$ such that $fx \in \theta x$ for every point $x \in X$, and $f^{-1}U$ is an F_σ-set for every open set U in Y.

§ 4. X-closed mappings and projections of Borel sets.

Theorem 11. Let the mapping $\theta : X \to F(Y)$ be X-closed, where Y is a complete metric space. If the space X is perfectly normal, then there exists a single-valued mapping $f : X \to Y$ such that $fx \in \theta x$ for every point $x \in X$, and $f^{-1}U$ is an F_σ-set for every open set U in Y.

From Proposition 1 and Theorem 11 it follows:

Corollary 2. Let $\theta : X \to C(Y)$ be an upper semicontinuous mapping of a perfectly normal space X into a metric space Y. Then there exists a single-valued mapping $f : X \to Y$ such that $fx \in \theta x$ and $f^{-1}U$ is an F_σ-set for every open set U in Y.

Theorem 11 together with the theorem of I. A. Vainshtein (see (3), Theorem 5) makes it possible to extend one result of A. D. Taimanov to arbitrary metric spaces:

Theorem 12. Let $f : X \to Y$ be a perfect mapping of a metric space X onto a space Y. If the space X is an absolute Borel set of class $\leq \alpha$, then the space Y is also an absolute Borel set of class $\leq 1 + \alpha$ for $\alpha < \omega_0$ and $\leq \alpha$ if $\alpha \geq \omega_0$.

However, Michael' s theorem (see (7), Theorem 1.1) and Theorem 12 allow us to assert:

Theorem 13. Let $f : X \to Y$ be an inductively open mapping of a metric space X onto a metric space Y, with complete point-preimages (in the metric given on X). If the space X is an absolute Borel set of class $\leq \alpha$, then the space Y is also an absolute Borel set of class $\leq 1 + \alpha$ for $\alpha < \omega_0$ and $\leq \alpha$ if $\alpha \geq \omega_0$.

* A mapping $f : X \to Y$ is inductively open if there exists a subspace $X_0 \subset X$ such that $fX_0 = Y$ and the mapping $f_1 = f|X_0$ is open (on X_0) (see (^{1})).

From a preprint kindly sent by R. Engelking I have learned the following result of his: let $f : X \to F(Y)$ be an upper semicontinuous mapping of a perfectly normal paracompact space X into a complete metric space Y. If fx is separable for every point $x \in X$, then there exists a single-valued mapping $g : X \to Y$ such that $gx \in fx$ for every point $x \in X$, and $g^{-1}U$ is an F_σ-set for every set U open in Y. This prompted me to prove the following assertion.

Theorem 14. Let $\theta : X \to F(Y)$ be an upper semicontinuous mapping of a perfectly normal and weakly paracompact space X into a complete metric space Y, where θx is separable for every point $x \in X$. Then there exists a single-valued mapping $f : X \to Y$ such that $fx \in \theta x$ for every point $x \in X$, and $f^{-1}U$ is an F_σ-set for every set U open in Y.

§ 5. **Remarks on ρ-continuous mappings.**

A mapping $\theta : X \to F(Y)$, where (Y, ρ) is a metric space, is ρ-continuous if for every point $x \in X$ and every $\varepsilon > 0$ there exists Ox such that

$$\min\{r \mid O(\theta x, r) \supset \theta z,\ O(\theta z, r) \supset \theta x\} < \varepsilon$$

for every point $z \in Ox$. From Theorem 10 it follows:

Theorem 15. Let $\theta : X \to F(Y)$ be a ρ-continuous mapping, where Y is a complete metric space. Then there exists a single-valued mapping $f : X \to Y$ such that $fx \in \theta x$ for every point $x \in X$, and $f^{-1}U$ is an F_σ-set for every set U open in Y.

Two theorems of E. Michael (see Theorem 1.1 of (^{7}) and Theorem 2 of (^{6})) and the factorization theorem from (^{2}) make it possible to obtain the theorem:

Theorem 16. Let $\theta : X \to F(Y)$ be a ρ-continuous mapping of a normal space X into a complete metric space Y. Then there exist mappings $\varphi : X \to C(Y)$ and $\psi : X \to C(Y)$ such that:
a) $\varphi x \subset \psi x \subset \theta x$ for every point $x \in X$;
b) φ is lower semicontinuous;
c) ψ is upper semicontinuous;
d) if $\dim X = 0$, then the sets φx and ψx are singletons for every point $x \in X$.

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Note: Figure translations are in progress. See original paper for figures.

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