APPLICATION OF THE NEWTON DIAGRAM TO THE PROBLEM OF STABILITY OF PERIODIC SOLUTIONS

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Abstract

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APPLICATION OF THE NEWTON DIAGRAM TO THE PROBLEM OF STABILITY OF PERIODIC SOLUTIONS

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1. The question of Lyapunov stability of periodic solutions of nonautonomous quasilinear systems has been studied by a number of authors. In doing so, various restrictions were imposed on the matrix A of the linear part of the system, and these restrictions were used essentially in the investigations of these authors. The method proposed here for solving the problem is not connected with any restrictions on the matrix A and leads to new propositions on the stability of periodic solutions.

2. Consider the equation

$$dx/dt = Ax + \lambda F(t, x, \lambda), \tag{1}$$

where $x = (x_1, \dots, x_n)$ is the required vector of the Euclidean space E_n; $t \in [0, +\infty)$; A is a constant real matrix; λ is a scalar parameter; $F = (F_1, \dots, F_n)$ is a vector-function, ω-periodic in t, continuous in the aggregate of the variables

$$(t, x, \lambda) \in [0, +\infty) \times G \times (0, \lambda_0)$$

(G is some domain of the space E_n, $\lambda_0 > 0$) and analytic in $x \in G$ and $\lambda \in (0, \lambda_0)$. It is assumed that the coefficients of the series F_i $(i = 1, \dots, n)$ are real functions of the argument t.

Let $\tilde{x}(t, \lambda) = (\tilde{x}_1(t, \lambda), \dots, \tilde{x}_n(t, \lambda))$ be a real ω-periodic solution of the Poincaré problem for equation (1). We shall clarify the question of its stability. Suppose that the components of the vector $\tilde{x}(t, \lambda)$ have the form

$$\tilde{x}_i(t, \mu) = \tilde{x}_i^0(t) + \sum_{k=1}^{\infty} x_{ik}(t)\mu^k, \tag{2}$$

where $\mu^{1/p}$ (p is a natural number and $\lambda^{1/p}$ is the arithmetic value of this root) and $\{\tilde{x}_i^0(t)\}_1^n$ is an ω-periodic solution of the generating equation $dx^0/dt = Ax^0$.

Let us note that in the nondegenerate case (^{1}) the number of all solutions of the Poincaré problem is finite and each of them is representable in the form (2).

By the shift $x = \tilde{x} + y$ we reduce the problem posed to the problem of stability of the trivial solution $y \equiv 0$ of the equation

$$\frac{dy}{dt} = \left(A + \sum_{k=0}^{\infty} B_k(t)\mu^{p+k} \right) y + \mu^p \Phi(t, y, \mu), \tag{3}$$

where

$$B_j(t) = \frac{1}{j!} \left(\frac{\partial^j F(t, \tilde{x} + y, \mu^p)}{\partial y \, \partial \mu^j} \right) \Bigg|_{y=0,\ \mu=0} \qquad (j = 0, 1, \dots)$$

$$\Phi(t, y, \mu) = \sum_{n=2}^{\infty} \frac{1}{n!} \left(\frac{\partial^n F(t, \tilde{x} + y, \mu^p)}{\partial y^n} \right) \Bigg|_{y=0} y^n.$$

We shall investigate, in the first approximation, the stability of the trivial solution of equation (3). Let $\rho_i(\mu)$ $(i = 1, \dots, n)$ be the multipliers of the equations

$$\frac{dy}{dt} = \left(A + \sum_{k=0}^{\infty} B_k(t)\mu^{p+k} \right) y. \tag{4}$$

All of them are continuous functions in some right semi-neighborhood $[0, \mu_0)$ of the point $\mu = 0$ and can be represented in the form

$$\rho_k(\mu) = e^{\gamma_k \omega} + \sigma_k(\mu) \qquad (k = 1, \dots, n), \tag{5}$$

where γ_k are the eigenvalues of the matrix A, and $\sigma_k(0) = 0$. In view of this we arrive at the following conclusion.

If $\operatorname{Re}\gamma > 0$ (respectively, $\operatorname{Re}\gamma < 0$) for some eigenvalue γ of the matrix A, then the multiplier $\rho(\mu)$ of equation (4) corresponding to this eigenvalue satisfies, in some right semi-neighborhood of the point $\mu = 0$, the inequality $|\rho(\mu)| > 1$ (respectively $|\rho(\mu)| < 1$). Hence, in particular, on the basis of the known Lyapunov theorem it follows that if at least one eigenvalue γ of the matrix A has positive real part, then the trivial solution of equation (3) is unstable. In connection with this, in what follows it is assumed that the eigenvalues γ_k of the matrix A satisfy the inequalities $\operatorname{Re}\gamma_k \leq 0$ $(k = 1, \dots, n)$; moreover, in order to decide whether the $|\rho_k(\mu)|$ will be greater or less than one, it remains only to investigate the additions $\sigma(\mu)$ to those $\rho(0) = e^{\gamma\omega}$ which correspond to the critical eigenvalues γ (i.e., zero and purely imaginary).

3. To investigate the indicated additions we compose the equation (see (5))

$$[M(\mu) - (\rho(0) + \sigma)E]z = 0, \tag{6}$$

where $M(\mu)$ is the monodromy matrix of equation (4); E is the identity matrix, and $z = (z_1, \dots, z_n)$ is a column vector of the n-dimensional Euclidean space E_n. We note that in the case under consideration the matrix $M(\mu)$ is represented in the form of a convergent series

$$M(\mu) = M_0 + \sum_{k=0}^{\infty} M_{p+k}\mu^{p+k}, \tag{7}$$

where $M_0 = e^{A\omega}$, and all the remaining coefficients are determined by a known recurrent process (see, for example, (2), pp. 227-228). By virtue of (7), equation (6) takes the form

$$Cz = -\sum_{k=0}^{\infty} M_{p+k}\mu^{p+k}z + \sigma z, \qquad \text{where } C = M_0 - \rho(0)E. \tag{6'}$$

We investigate the additions σ to those $\rho(0)$ which correspond to resonant critical eigenvalues γ_k of the matrix A, i.e., eigenvalues of the form

$$\gamma_k = \pm 2\pi i p_k/\omega \qquad (k = 1, \dots, \bar{k}), \quad \gamma_0 = 0, \tag{8}$$

where p_k are natural numbers.

By virtue of (5), to the entire set (8) there corresponds only the single eigenvalue $\rho(0) = 1$ of the matrix M_0. Denote by l the multiplicity of this eigenvalue and by m the dimension of the subspace of solutions of the equation $Cz = 0$, where $C = M_0 - E$.

We note that, according to Theorems 9 and 8 of Ch. IV from (3),

$$m = m_0 + 2\sum_{h=1}^{\bar{k}} m_k,$$

where m_0 is the number of all elementary divisors of the matrix A corresponding to the eigenvalue $\gamma_0 = 0$, and m_k is the number of all elementary divisors of this matrix corresponding to each eigenvalue of the pair $\pm 2\pi i p_k/\omega$. Moreover, $l \geq m$, and the indicated relation becomes an equality only in the case when all the indicated elementary divisors are simple. We also note that the approach to the stability problem considered here is not connected with any restrictions on the numbers m and l.

Let $\varphi_k = (a_1^{(k)}, \dots, a_n^{(k)})$ and $\psi_k = (b_1^{(k)}, \dots, b_n^{(k)})$ $(k = 1, \dots, m)$ be orthonormal bases, respectively, for the subspaces of solutions of the equations $Cz = 0$ and $C^*u = 0$, where C^* is the matrix adjoint to C.* Then equation $(6')$ can be written in the form

$$Dz = -\sum_{k=0}^{\infty} M_{p+k}\mu^{p+k} z + \sigma z - \sum_{j=1}^{m} \xi_j \psi_j, \tag{9}$$

where

$$D = C - N, \qquad N = \sum_{k=1}^{m} [\psi_k a_1^{(k)}, \dots, \psi_k a_n^{(k)}], \qquad \xi_j = \langle z, \varphi_j \rangle$$

are scalar products. It is established that the matrix D is invertible, whence, by the implicit function theorem, equation (9) has a unique local solution

$$z = \sum_{j=1}^{m} \left[c_{00}^{(j)} + c_{0p}^{(j)}\mu^p + c_{0,p+1}^{(j)}\mu^{p+1} + c_{10}^{(j)}\sigma + c_{20}^{(j)}\sigma^2 + c_{1p}^{(j)}\sigma\mu^p + \dots \right] \xi_j. \tag{10}$$

Substituting the series (10) into the solvability conditions for equation $(6')$

$$\left\langle \sigma z - \sum_{k=0}^{\infty} M_{p+k}\mu^{p+k} z,\ \psi_j \right\rangle = 0 \quad (j = 1, \dots, m),$$

we arrive at a homogeneous linear equation with respect to ξ,

$$G(\sigma, \mu)\xi = 0, \tag{11}$$

where ξ is the column vector $(\xi_1, \dots, \xi_m)$.

Note that $\rho = 1 + \sigma$ is an eigenvalue of the monodromy matrix if and only if equation (11) admits a nonzero solution, i.e., only under the condition

$$\det G(\sigma, \mu) = 0. \tag{12}$$

Equation (12) is represented in the form

$$\sum_{k} L_{k,0}\sigma^k + \sum_{k+n \geq m-1} L_{k,n+1}\sigma^k \mu^{p(n+1)} = 0, \tag{13}$$

where $L_{k,0} = 0$ for $k < l$ and $L_{l,0} \neq 0$.

4. To find all small solutions $\sigma(\mu)$ of equation (13), we apply the Newton diagram method (see $(^4)$), according to which all nontrivial small solutions of this equation are represented by convergent series

$$\sigma(\mu) = \zeta\mu^{r/s} + o(\mu^{r/s}), \qquad (r,s) = 1, \tag{14}$$

where r/s is the slope of the corresponding segment of the decreasing part of the diagram, and ζ is a root of the determining equation for this segment.

In what follows the following lemma is used.

Lemma. Let $s > 1$. Then, if the determining equation has a root with negative real part, it also has a root whose real part is nonnegative.

We shall call a multiplier $\rho(\mu)$ of equation (4) **critical** if $\rho(\mu) \equiv \rho(0)$, and **noncritical** otherwise.

Equation (4) has no critical multipliers $\rho(\mu) \equiv 1$ if and only if at least one of the coefficients $L_{0,n+1}$ of equation (13) is nonzero.

* In the case under consideration C is a real matrix.

From (5) and (14) it follows that all noncritical multipliers $\rho(\mu)$ corresponding to resonant eigenvalues (9) are represented in the form $\rho(\mu) = 1 + \zeta\mu^{r/s} + o(\mu^{r/s})$, and, for estimating $|\rho(\mu)|$ for small μ, one may restrict oneself to their approximate values

$$\tilde{\rho}(\mu) = 1 + \zeta\mu^{r/s}. \tag{15}$$

5. With the aid of the lemma and representation (15), for noncritical multipliers one establishes a number of assertions on the stability of the trivial solution of equation (3), of which we shall give the following.

Theorem 1. *If, in the resonant case, at least one root of the determining equation of some segment of the decreasing part of the diagram has a nonnegative real part, then the trivial solution of equation (3) is unstable.*

Theorem 2. *If, in the resonant case, for at least one segment of the decreasing part of the diagram the number $s > 1$, then the trivial solution of equation (3) is unstable.*

We give one sufficient condition for asymptotic stability. Let us single out a class R of differential equations of the form (3) for which all eigenvalues γ of the matrix A have nonpositive real parts, and those γ for which $\operatorname{Re}\gamma = 0$ are resonant and the equation (13) compiled for them has no trivial solutions.

Theorem 3. *For asymptotic stability of the trivial solution of equation (3), belonging to the class R, it is sufficient that the roots of the determining equations of all segments of the decreasing part of the diagram have negative real parts.*

6. The investigation of the stability of the trivial solution of equation (3) in the case when the matrix A has nonresonant critical eigenvalues does not lead to additional difficulties. Just as in the case of resonant eigenvalues, in order to determine the corrections σ we arrive at an equation of the form (13), which is investigated by means of the Newton diagram. For small μ, in estimating the modulus of a noncritical multiplier $\rho(\mu) = \rho(0) + \zeta\mu^{r/s} + o(\mu^{r/s})$ $\big(|\rho(0)| = 1,\ \mu^{r/s} > 0,\ \zeta$ is a root of the determining equation), one may restrict oneself to its approximate value $\tilde{\rho}(\mu) = \rho(0) + \zeta\mu^{r/s}$.

Let us make the following useful remark. The inequalities $|\tilde{\rho}(\mu)|^2 > 1$ and $|\tilde{\rho}(\mu)|^2 < 1$ take respectively the form

$$(\alpha a + \beta b)/\mu^{r/s} + (a^2 + b^2)/2 > 0, \quad (\alpha a + \beta b)/\mu^{r/s} + (a^2 + b^2)/2 < 0, \qquad (16)$$

where $\alpha = \operatorname{Re}\rho(0)$, $\beta = \operatorname{Im}\rho(0)$, $a = \operatorname{Re}\zeta$, $b = \operatorname{Im}\zeta$. Since $\mu^{r/s} > 0$, for small μ and when $\alpha a + \beta b \neq 0$, inequality (16) is equivalent respectively to the inequality

$$\alpha a + \beta b > 0, \quad \alpha a + \beta b < 0. \qquad (17)$$

With the aid of inequalities (16) or (17), various criteria are established both for instability and for asymptotic stability.

7. The method of investigation is also applicable in the case when $A = A(t)$ is a continuous ω-periodic matrix; moreover, if the equation $dx/dt = A(t)x$ is integrated in closed form, then this method is easily implemented.

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