GENERAL SOLUTIONS IN THE THEORY OF MAGNETOHYDRODYNAMIC FLOWS IN THE HYPERSONIC APPROXIMATION

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Abstract

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GENERAL SOLUTIONS IN THE THEORY OF MAGNETOHYDRODYNAMIC FLOWS IN THE HYPERSONIC APPROXIMATION

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General solutions have been obtained and the corresponding Cauchy problems have been solved for spatial nonstationary and stationary flows of an anisotropically conducting inviscid gas in a magnetic field and in the absence of an electric field, for Mach numbers $M \gg 1$, when the pressure gradient may be neglected in the equations of motion.

The equations of motion of a conducting inviscid gas have the well-known form (notation is standard)

$$\rho(\partial/\partial t + \mathbf{v}\nabla)\mathbf{v} + \nabla p = [\mathbf{jB}]. \tag{1}$$

The Lorentz force $[\mathbf{jB}]$ is expressed in terms of the effective electric-field strength $\mathbf{E}^* = \mathbf{E} + [\mathbf{vB}]$ by means of the generalized Ohm' s law (see, for example, (^{1})):

$$[\mathbf{jB}] = \frac{\sigma_\perp B}{1+\delta_\perp^2}([\mathbf{E}_\perp^* \mathbf{b}] + \delta_\perp \mathbf{E}_\perp^*), \tag{2}$$

where $\mathbf{b} = \mathbf{B}/B$; $\sigma_\perp$ is the transverse conductivity; $\delta_\perp$ is the tangent of the angle between $\mathbf{j}_\perp$ and $\mathbf{E}_\perp^*$. Ion slip is taken into account by the dependence of $\sigma_\perp$ on B (^{1}). Without limiting generality, we shall take the magnetic field to be directed along the z-axis, namely: $\mathbf{B}(0,0,-B)$. Introducing the load parameters $K_x = E_x/v_{yB}$, $K_y = -E_y/v_{xB}$, in the hypersonic approximation, when $M \gg 1$ and the pressure gradient may be neglected, we obtain the system of equations (1), expanded with respect to $\mathbf{v}(u,v,w)$,

$$Du + \nu_1 u + \delta_\perp \nu_2 v = 0, \qquad Dv - \delta_\perp \nu_1 u + \nu_2 v = 0, \qquad Dw = 0, \tag{3}$$

where

$$D \equiv \partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z; \qquad \nu_1 = \nu(1 - K_y); \qquad \nu_2 =$$

$$= \nu(1 - K_x); \qquad \nu = \frac{1}{1+\delta_\perp^2}\frac{\sigma_\perp B^2}{\rho}$$

is the frequency of magnetohydrodynamic interaction.

The simplicity of the hypersonic approximation adopted here is due to the fact that, for given $\nu_{1,2}$ and $\delta_\perp$, the equations of motion (3) form a closed system for u, v, w, which is solved independently of the equations of continuity, energy, and Maxwell' s equations. For simplicity we shall regard the quantities $\nu, K_x, K_y, \delta_\perp$ as constant. Then system (3) admits a broad class of general solutions, the derivation of which constitutes the content of this work. We note that certain particular problems with zero pressure gradient were considered in a series of papers (^{2}). In what follows it is assumed that the electric field, if present, is only along the z-axis, so that $\nu_1 = \nu_2 = \nu$. In addition, the no-load regime along one of the axes x or y will be considered, when one of the quantities ν_1 or ν_2 is equal to 0. In such regimes the condition of constancy of the parameter ν can be reconciled with the need to satisfy the remaining equations (in particular, the current-continuity equation).

In the simplest nonstationary one-dimensional case, the single equation (3)

$$u_t + uu_x + \nu_1 u = 0 \tag{4}$$

(the subscripts denote differentiation) has the first integrals

$$e^{\nu_1 t}u = C_1, \qquad u + \nu_1 x = C_2$$

and the general solution with an arbitrary function f

$$u + \nu_1 x = f(e^{\nu_1 t}u). \tag{5}$$

The Cauchy problem for equation (4) is as follows: find $u(x,t)$ such that $u(x,0) = u_0(x)$. Let the inverse dependence of x on u_0 have the form $x = \varphi(u_0)$. Then from (5) at $t = 0$ we obtain $u_0 + \nu_1\varphi(u_0) = f(u_0)$, or $e^{\nu_1 t}u + \nu_1\varphi(e^{\nu_1 t}u) = f(e^{\nu_1 t}u) = u + \nu_1 x$, so that $\varphi(e^{\nu_1 t}u) = x - \dfrac{u}{\nu_1}(e^{\nu_1 t} - 1)$, and, inverting the last expression, we obtain the final result

$$u = e^{-\nu_1 t}u_0\left[x - \frac{u}{\nu_1}\,(e^{\nu_1 t} - 1)\right]. \tag{6}$$

For $\nu_1 \to 0$ one obtains $u = u_0(x - ut)$, as should be the case. The boundary-value problem is solved analogously, when it is required to find $u(x,t)$ such that $u(0,t) = u_0(t)$. The solution has the form

$$u = u_0\left[t - \frac{1}{\nu_1}\ln\left(1 + \frac{\nu_1 x}{u}\right)\right] - \nu_1 x. \tag{7}$$

A solution analogous to (6) also holds in the general case for $\delta_\perp = 0$, namely:

$$u = e^{-\nu_1 t}u_0(\xi,\eta,\zeta), \quad v = e^{-\nu_2 t}v_0(\xi,\eta,\zeta), \quad w = w_0(\xi,\eta,\zeta), \tag{8}$$

where $\xi = x - \dfrac{u}{\nu_1}(e^{\nu_1 t} - 1)$, $\eta = y - \dfrac{v}{\nu_2}(e^{\nu_2 t} - 1)$, $\zeta = z - wt$. The method of solution consists in introducing new functions $\varepsilon = v/u$, $\delta = w/u$, with their subsequent use as independent variables.

Let us consider in more detail the nonstationary plane case $\nu_1 = \nu_2 = \nu$ and $\delta_\perp \neq 0$. The equations have the form

$$u_t + uu_x + vu_y + \nu(u + \delta_\perp v) = 0,$$

$$v_t + uv_x + vv_y + \nu(-\delta_\perp u + v) = 0. \tag{9}$$

Introducing $\varepsilon = v/u$, from the combination (9) we obtain the equation for ε

$$\varepsilon_t + u(\varepsilon_x + \varepsilon\varepsilon_y) - \nu\delta_\perp(1 + \varepsilon^2) = 0.$$

Passing from the variables t, x, y to the variables ε, x, y, we obtain from the combination of these equations the equation

$$\frac{1}{u}\nu\delta_\perp(1 + \varepsilon^2)u_\varepsilon + u_x + \varepsilon u_y + \nu(1 + \delta_\perp\varepsilon) = 0,$$

whose first integrals have the form

$$\exp(\operatorname{arc\,tg}\varepsilon/\delta_\perp)\sqrt{1 + \varepsilon^2}\,u = C_1,$$

$$\nu x + \frac{1}{1 + \delta_\perp^2}(u - \delta_\perp v) = C_2, \qquad \nu y + \frac{1}{1 + \delta_\perp^2}(\delta_\perp u + v) = C_3.$$

If one passes from the variables t, x, y to the variables t, ε, y or t, x, ε, then we obtain the last integral

$$\nu t - \operatorname{arc\,tg} \varepsilon/\delta_\perp = C_4.$$

As a result, the general solution, containing two arbitrary functions $f_{1,2}$, has the form

$$u + \nu(x + \delta_\perp y) = f_1(\alpha, \beta), \qquad v + \nu(-\delta_\perp x + y) = f_2(\alpha, \beta), \tag{10}$$

where $\alpha = e^{\nu t}\sqrt{1+\varepsilon^2}\, u$, $\beta = \operatorname{arc\,tg} \varepsilon - \delta_\perp \nu t$.

In (10) it is convenient to introduce the total velocity $V = \sqrt{u^2 + v^2}$ and the angle $\theta = \operatorname{arc\,tg}(v/u)$. For equations (9) one can pose the Cauchy problem: find $V(t, x, y)$ and $\theta(t, x, y)$ such that $V(0, x, y) = V_0(x, y)$ and $\theta(0, x, y) = \theta_0(x, y)$. Let the inverse dependences of x, y on V_0, θ_0 have the form $x = \varphi_1(V_0, \theta_0)$, $y = \varphi_2(V_0, \theta_0)$. Then from (10), for $t = 0$, we obtain

$$V_0^2 = \{f_1(V_0, \theta_0) - \nu[\varphi_1(V_0, \theta_0) + \delta_\perp \varphi_2(V_0, \theta_0)]\}^2 +$$

$$+\{f_2(V_0, \theta_0) - \nu[-\delta_\perp \varphi_1(V_0, \theta_0) + \varphi_2(V_0, \theta_0)]\}^2,$$

$$\theta_0 = \operatorname{arc\,tg} \frac{f_2(V_0, \theta_0) - \nu[-\delta_\perp \varphi_1(V_0, \theta_0) + \varphi_2(V_0, \theta_0)]}{f_1(V_0, \theta_0) - \nu[\varphi_1(V_0, \theta_0) + \delta_\perp \varphi_2(V_0, \theta_0)]}.$$

Replacing V_0 by $\alpha = e^{\nu t} V$ and θ_0 by $\beta = \theta - \delta_\perp \nu t$, with the subsequent replacements of $f_{1,2}$ according to (10), and solving the equations with respect to $\varphi_1(\alpha, \beta)$ and $\varphi_2(\alpha, \beta)$, after inverting the latter functions we obtain expressions for V and θ, i.e. the desired solution of the Cauchy problem in explicit form.

The general case, when $\nu_1 \neq \nu_2$ and $\delta_\perp \neq 0$, in principle can also be solved, but because of the cumbersomeness of the results it is not given here.

Consider the simplest steady plane case $\nu_1 = \nu_2 = \nu$ and $\delta_\perp = 0$. The equations have the form

$$u u_x + v u_y + \nu u = 0, \qquad u v_x + v v_y + \nu v = 0. \tag{11}$$

Proceeding analogously, we obtain the general solution

$$u + \nu x = f_1(\varepsilon), \qquad v + \nu y = f_2(\varepsilon). \tag{12}$$

From (12) it follows that

$$u + \nu x = f_3(x - y/\varepsilon), \qquad v + \nu y = f_4(x - y/\varepsilon).$$

Let $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$. Then the solution of such a Cauchy problem has the form

$$u = (1 + \nu y/v)^{-1} u_0(x - y/\varepsilon), \qquad v = (1 + \nu y/v)^{-1} v_0(x - y/\varepsilon). \tag{13}$$

Analogously, the solution of the Cauchy problem $u(0,y) = u_0(y)$ and $v(0,y) = v_0(y)$ has the form

$$u = (1 + \nu x/u)^{-1} u_0(y - x\varepsilon)/u)^{-1}, \qquad v = (1 + \nu x/v_0(y - x\varepsilon). \tag{14}$$

A somewhat more general steady plane case, when $\nu_1 \neq \nu_2$ and $\delta_\perp = 0$, turns out to be considerably more complicated. The general solution, as is not difficult to verify, has the form

$$u + \nu_1 x = f_1(vu^{-\mu}), \qquad v + \nu_2 y = f_2(vu^{-\mu}), \tag{15}$$

where $\mu = \nu_2/\nu_1 \neq 1$. Let $u(x,0) = u_{01}(x)$ and $v(x,0) = v_{01}(x)$, so that $v_{01} u_{01}^{-\mu} = \psi_1(x)$, where ψ_1 is a prescribed function of x. Then the solution of such a particular Cauchy problem has the form

$$vu^{-\mu} = \psi_1\{x - [(1 + \nu_2 y/v)^{1/\mu} - 1]u/\nu_1\}. \tag{16}$$

Next, let $u(0,y) = u_{02}(y)$ and $v(0,y) = v_{02}(y)$, so that $u_{02} v_{02}^{-1/\mu} = \psi_2(y)$, where ψ_2 is a prescribed function of y. Then the solution of such a particular Cauchy problem has the form

$$uv^{-1/\mu} = \psi_2\{y - [(1 + \nu_1 x/u)^{\mu} - 1]v/\nu_2\}. \tag{17}$$

As a result, one can solve the general Cauchy problem when $vu - u$ (or $uv^{-1/\mu}$) is prescribed at $x = 0$ and $y = 0$.

In the limiting situation when, for example, $\nu_2 \to 0$, i.e. $\mu \to 0$, solution (15) in the equivalent form

$$u + \nu_1 x = f_1(vu^{-\mu}), \qquad -\frac{v(u^{-\mu} - 1)}{\mu} + \nu_1 y = f_2(vu^{-\mu})$$

gives

$$u + \nu_1 x = f_1(v), \qquad v \ln u + \nu_1 y = f_2(v), \tag{18}$$

so that the solution of the particular Cauchy problem $v(x,0) = v_{01}(x)$ has the form

$$v = v_{01}\left[x - \frac{u}{\nu_1}\left(e^{\nu_1 y/v} - 1\right)\right]. \tag{19}$$

Similarly, the solution of the particular Cauchy problem $v(0, y) = v_{02}(y)$ has the form

$$v = v_{02}\left[y - \frac{v}{\nu_1}\ln\left(1 + \frac{\nu_1 x}{u}\right)\right]. \tag{20}$$

As a result, one can solve the general Cauchy problem when v is prescribed on $x = 0$ and $y = 0$.

The solution of the Cauchy problem in the stationary plane case, when $\nu_1 \neq \nu_2 = \nu$ and $\delta_\perp \neq 0$, is possible only in implicit form for the inverse functions. This solution, as well as more general solutions, is not given here.

It is not difficult to reproduce the general solutions obtained here also in other coordinate systems (cylindrical, spherical, etc.).

The fact that in the nonstationary case the solutions of the Cauchy problem are simpler than in the stationary case is a consequence of the structure of the original equations (3), in which differentiation with respect to time is distinguished. In other words, the asymmetry in the formulation of the boundary-value problem in the stationary case does not correspond to symmetric differentiation with respect to the spatial coordinates in the original equations. However, one can introduce a parametric representation of the coordinates. Then the formulation of the boundary-value problem will be symmetric.

In the absence of a magnetic field, the solutions admit discontinuities. Imposition of a magnetic field leads to smoothing of the initial distributions, so that the class of initial conditions that do not lead to discontinuities is enlarged. In the simplest nonstationary one-dimensional case, a sufficient condition for the absence of discontinuities is the condition $-du_0(x)/dx < \nu_1$.

In conclusion, we note that analogous general solutions can also be obtained in the presence of an electric field satisfying the condition $\mathbf{E}/B = v^0 = \text{const}$, which is fulfilled in constant fields. Thus, for example, solution (4), upon replacing $\nu_1 u$ by $\nu_1(u - u^0)$, where $u^0 = \text{const}$, has the form

$$u = e^{-\nu_1 t} u_0\left[x - \frac{u - u^0}{\nu}\left(e^{\nu_1 t} - 1\right) - u^0 t\right] + u_0\left(1 - e^{-\nu_1 t}\right). \tag{21}$$

This expression for $u^0 = 0$ coincides with (6).

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Note: Figure translations are in progress. See original paper for figures.