ON BOUNDED SOLUTIONS OF THE RIEMANN BOUNDARY-VALUE PROBLEM WITH AN INFINITE INDEX OF POWER ORDER

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Abstract

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1°. Let us take the domain D, obtained from the complex z-plane by removing the ray $L = (1 \le t \le \infty)$. In this domain we consider the Riemann boundary-value problem with infinite index

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \qquad (1 < t < \infty) \tag{1}$$

under the following assumptions:

$$1. \quad \arg G(t) = 2\pi\varphi(t)t^\rho, \quad -1 < \varphi(1) \le 0, \quad \varphi(\infty) \ne 0, \quad \rho > 0, \tag{2}$$

where $\varphi(t)$ is Hölder continuous, or $\varphi(t) \in H(\mu)$, i.e.

$$|\varphi(t_1) - \varphi(t_2)| < A|1/t_1 - 1/t_2|^\mu, \qquad A, \mu = \text{const.} \tag{3}$$

$$2. \quad \ln|G(t)|, \quad g(t) \in H(\lambda), \qquad 0 < \lambda \le 1. \tag{4}$$

$$3. \quad g(1) = g(\infty) = 0. \tag{5}$$

In addition, the Hölder exponent μ is subject to the additional restriction

$$(2\rho - 1)/(2\rho + 1) < \mu \le 1, \tag{6}$$

if the homogeneous problem is considered, i.e. when $g(t) \equiv 0$, and

$$\rho/(\rho + 1) < \mu \le 1, \tag{7}$$

if the problem is nonhomogeneous. We note that condition (6) is automatically fulfilled when $\rho \leq 1/2$.

We shall solve problem (1) in the class B, consisting of functions analytic and bounded in the domain D. In papers ([4,5]) problem (1) was solved in a considerably narrower class, namely, solutions were sought that are bounded and have completely regular growth of order $\sigma < \min(\rho, 1/2)$. Solutions in the class B for the case $\rho < 1/2$ were considered in paper ([3]); in this case condition (7) was replaced by the more restrictive condition $\mu > 0$. We note that for $\rho \geq 1$ the latter condition would be equivalent to $\varphi(t) = \text{const}$ ([1], p. 22).

Here only the case with plus-infinite index is considered, i.e. with $\varphi(\infty) > 0$ in relation (2). The problem with minus-infinite index was investigated in ([5]).

2°. We first dwell on the homogeneous problem

$$\Psi^+(t) = G(t)\Psi^-(t). \tag{8}$$

Theorem 1. *The general solution in the class B of the homogeneous boundary-value problem (8) is expressed by the formula*

$$\Psi(z) = Cz^m \prod_{n=1}^{n_0} \frac{1 - z/z_n}{1 - z/r_n} \exp\left[\frac{z}{2\pi i}\int_0^\infty \frac{\ln G(x) - 2\pi i \tilde{n}_\Psi(x)}{x(x-z)}\,dx\right] \tag{9}$$

$$(n_0 \leq \infty, \quad C = \text{const}, \quad z_n = r_n e^{i\theta_n} \neq 0),$$

where $m \geqslant 0$ is an integer, and the sequence $\{z_n\}$, coinciding with the set of nonzero roots of $\Psi(z)$, satisfies the following requirements:

1. $0 < r_1 \leqslant r_2 \leqslant \cdots$
2. The series converges
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{r_n}} \sin\frac{\theta_n}{2}, \qquad 0 \leqslant \theta_n < 2\pi.$$
3. The function $\tilde{n}_\Psi(r)$, equal to the number of points z_n in the ring $0 < |z| < r$, is such that there exists a finite nonnegative limit
$$\lim_{r\to\infty} \frac{1}{4\sqrt{r}} \int_0^r \frac{dt}{t} \int_0^t \left[\frac{1}{2\pi}\arg G(x) - \tilde{n}_\Psi(x)\right]\frac{dx}{x} = \gamma \geqslant 0.$$
4. The integral converges
$$\int_0^\infty [\ln G(x) - 2\pi i \tilde{n}_\Psi(x)] x^{-2}\,dx.$$

5. On the contour L, the boundary function $\Psi^+(t)$ is bounded for sufficiently large t, or, what is the same, the asymptotic estimate holds

$$m\ln t+\sum_{n=1}^{\infty}\ln\left|\frac{t-z_n}{t-r_n}\right|+\frac{t}{2\pi}\int_0^{\infty}\frac{\arg G(x)-2\pi\tilde{n}_\Psi(x)}{x(x-t)}\,dx<\text{const.} \quad (10)$$

Remark. Formula (9) can be written in the form

$$\Psi(z)=F(z)\exp\left[\frac{z^{q+1}}{2\pi i}\int_0^{\infty}\frac{\ln G(x)}{x^{q+1}(x-z)}\,dx\right], \qquad q=[\rho]^*,$$

where $F(z)$ is an entire function of order ρ, representable in the form

$$F(z)=cz^m e^{P_q(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)\exp\left[\frac{z}{r_n}+\cdots+\frac{1}{q}\left(\frac{z}{r_n}\right)^q\right].$$

The set of zeros of the function $F(z)$ coincides with the set of zeros of the solution $\Psi(z)$ itself and is subject to conditions 1–5, while

$$P_q(z)=\sum_{k=1}^{q}\frac{z^k}{2\pi i}\int_0^{\infty}[\ln G(x)-2\pi i\tilde{n}_\Psi(x)]x^{-k-1}\,dx.$$

Following B. Ya. Levin ([2], p. 120), we shall call a C^0-set an arbitrary set of points of the complex plane which can be covered by disks $|z-z_n|<r_n$, $n=1,2,\dots$, subject to the condition

$$\lim_{r\to\infty}\frac{1}{r}\sum_{|z_n|<r}r_n=0.$$

Theorem 2. If $\Psi(z)$ is a solution of class B of the homogeneous problem (8), then for all z not belonging to some C^0-set the asymptotic equality holds

$$\ln|\Psi(re^{i\theta})|=-\pi\gamma\sqrt{r}\sin\theta/2 \qquad (0\leqslant\theta\leqslant 2\pi),$$

where γ is the limit defined in Theorem 1.

Problem (8) has an infinite set of linearly independent solutions of class B. As the simplest example of such a solution one may

* $[a]$ denotes the integer part of the real number a.

to the following:

$$\Psi(z)=\exp\left[\frac{z}{2\pi i}\int_0^{\infty}\frac{\ln G(x)-2\pi i n_\Psi(x)}{x(x-z)}\,dx\right]P_\nu(z),$$

$$n_\Psi(t) = \max\left\{0, \left[\max_{0\le x\le t}\{\varphi(x)x^\rho - \nu\}\right]\right\}, \tag{11}$$

where ν is an arbitrary positive number, and $P_\nu(z)$ is a polynomial of degree not exceeding $[\nu]$. All zeros of the solutions (11), other than the zeros of $P_\nu(z)$, are simple and are located on the contour L at the discontinuity points of the function $n_\Psi(t)$.

The following theorem gives a uniqueness condition for the solution of the homogeneous problem.

Theorem 3. *Two solutions of problem (8), $\Psi_1(z)$ and $\Psi_2(z)$, belonging to the class B, and having common roots of the same multiplicity, can differ only by a constant factor: $\Psi_1(z) = C\Psi_2(z)$.*

3°. We shall formulate results pertaining to the nonhomogeneous problem.

Theorem 4. *The general solution in the class B of the nonhomogeneous Riemann problem (1) is expressed by the formula*

$$\Phi(z) = \Psi(z) + \frac{\Psi_0(z)}{2\pi i}\int_1^\infty \frac{g(x)\,dx}{\Psi_0^+(x)(x-z)}, \tag{12}$$

in which $\Psi(z)$ is determined by equality (9), while $\Psi_0(z)$ is a particular solution of the homogeneous problem (8) of the form

$$\Psi_0(z) = \exp\left[\frac{z}{2\pi i}\int_0^\infty \frac{\ln G(x) - 2\pi i n_{\Psi_0}(x)}{x(x-z)}\,dx\right]\prod_{n=1}^\infty\left(1-\frac{z}{r_n}e^{-ir_n^{-\rho}}\right)\left(1-\frac{z}{r_n}\right)^{-1}, \tag{13}$$

where the sequence $1 \le r_1 \le r_2 \le \cdots$ is determined by its counting function $n_{\Psi_0}(r)$ according to the formula

$$n_{\Psi_0}(r) = \max\left\{0, \left[\max_{1\le x\le r}\left\{\varphi(x)x^\rho + \frac{1}{2}\right\}\right]\right\}.$$

Remark. The function $\Psi_0(z)$ may be taken in the simpler form:

$$\Psi_0(z) = \exp\left[\frac{z}{2\pi i}\int_0^\infty \frac{\ln G(x) - 2\pi i n_{\Psi_0}(x)}{x(x-z)}\,dx\right].$$

This, however, is inconvenient because the Cauchy-type integral in (12) must then be understood infinitely many times in the sense of the Cauchy principal value, since its density $g(x)/\Psi_0^+(x)$ has, generally speaking, a countable set of simple poles.

4°. Conditions (6) and (7) are essential for solvability in the class B of problems (1) and (8), respectively.

Theorem 5. *The homogeneous Riemann boundary-value problem with plus-infinite index*

$$\Psi^{+}(t) = G_0(t)\Psi^{-}(t), \qquad 1 < t < \infty, \tag{14}$$

where $\arg G_0(t) = 2\pi\varphi_0(t)$,

$$\varphi_0(t) = 1 - \frac{8\rho}{\rho - 1}\, t^{1/2-\rho} \sin\left(\frac{\pi}{2} t^{\rho-1/2}\right), \qquad |G_0(t)| = 1,$$

$$\frac{1}{2} < \rho < \infty,$$

has no bounded solutions.

Conditions (2), (4) are here, obviously, satisfied, but condition (6) is violated, since, putting

$$\tau_n = (4n)^{2/(2\rho-1)}, \qquad t_n = (4n+1)^{2/(2\rho-1)},$$

we easily verify that for $n = 1, 2, ...$

$$|\varphi_0(t_n) - \varphi_0(\tau_n)| > \frac{2\rho - 1}{4}\left|\frac{1}{t_n} - \frac{1}{\tau_n}\right|^{(2\rho-1)/(2\rho+1)}, \tag{15}$$

and therefore $\varphi_0 \notin H(\mu)$ for any $\mu > (2\rho - 1)/(2\rho + 1)$. On the other

On the other hand, $\varphi_0 \in H((2\rho-1)/(2\rho+1))$, which is not difficult to establish with the aid of the following lemma.

Lemma 1. Let $a(t)$ and $\beta(t)$ be given for $0 < a \le t < \infty$, and let $a(t) \in H(\nu)$, $0 < \nu \le 1$, $a(\infty) = 0$, $|\beta'(t)| < Ct^{\mu-1}$. Then $a(t)\beta(t) \in H(\nu/(\mu+1))$.

In the proof of Theorem 5 the following is used.

Lemma 2. If the homogeneous problem (8), in which $G(t)$ satisfies conditions (2)–(4), has a bounded solution $\Psi(z)$, then there exists a finite limit

$$\lim_{\substack{r\to\infty \\ r\in E}} \omega_\Psi(r) = c \ge 0, \qquad \omega_\Psi(t) = [\varphi(t)t^{\rho} - \tilde{n}_\Psi(t)]/\sqrt{t}, \tag{16}$$

where $\tilde{n}_\Psi(t)$ has the same meaning as in (9), and E is a certain set of zero relative measure ((2), p. 127).

If we now assume that problem (14) has some solution $\Psi^* \in B$, then (16) must hold for it. This, however, is impossible, since one can show that

$$\left|\omega_{\Psi^*}\left((4n+h_1)^{2/(2\rho-1)}\right) - \omega_{\Psi^*}\left((4n+h_2)^{2/(2\rho-1)}\right)\right| \geq \frac{2\rho}{\rho-1}, \qquad (17)$$

if n is sufficiently large and h_k are arbitrary numbers such that $0 < h_1 < h_2 < 1$, $h_2 - h_1 > 2/3$. The contradiction obtained proves Theorem 5.

More difficult to prove is

Theorem 6. The nonhomogeneous Riemann boundary-value problem with plus-infinite index

$$\Phi^+(t) = G_1(t)\Phi^-(t) + g_1(t), \qquad 1 < t < \infty,$$

where $\arg G_1(t) = \varphi_1(t)t^\rho$, $\varphi_1(t) = 1 - t^{\delta-\rho}\sin\left(\frac{\pi}{2}t^\delta\right)$, $|G_1(t)| \equiv 1$, $g_1(t) =$

$$= (t-1)t^{-1-\eta}e^{-it^\rho}, \qquad \rho > \delta > 0,$$

has no solutions in the class B.

The fulfillment of conditions (2)–(5) here is verified in the same way as in Theorem 5. With the aid of an inequality analogous to (17), one can verify that (7) does not hold; more precisely, $\varphi_1 \in H(\mu)$ for no $\mu > (\rho-\delta)/(\rho+1)$. On the other hand, by Lemma 2, $\varphi_1 \in H((\rho-\delta)/(\rho+1))$. Thus, for $\mu < \rho/(\rho+1)$ the nonhomogeneous problem may turn out to be unsolvable. The question of whether it is always solvable for $\mu = \rho/(\rho+1)$ remains open.

Remark. By constructing appropriate examples one could also confirm the fact that the fulfillment of conditions (6) and (7) is not necessary for the solvability of the corresponding problems.

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Note: Figure translations are in progress. See original paper for figures.