SOME CONSEQUENCES OF THE HYPOTHESIS ON THE UNCOUNTABILITY OF THE SET OF CONSTRUCTIVE REAL NUMBERS

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Abstract

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SOME CONSEQUENCES OF THE HYPOTHESIS ON THE UNCOUNTABILITY OF THE SET OF CONSTRUCTIVE REAL NUMBERS

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The theorem stated and its proof are set out here within the framework of naive set theory. However, the arguments used are far from antinomies and can be formalized. They can be recast, for example, in the Zermelo–Fraenkel system (see ([1]), notation ZF) or in Gödel' s system (see ([3])). The concepts used are defined in papers ([2–4]). All the theorems to which we refer are given in those same papers.

Theorem. *If the set of constructive real numbers is uncountable, then one can construct an uncountable set of type CA without a perfect subset; if there exists a perfect set of positive measure all of whose elements are constructive real numbers, then one can construct a nonmeasurable set of type B_2.*

Proof. Denote, as is often done ([3,4]), the restriction of the Gödel function $F(a)$ (see ([3])) to the set of all countable ordinals by the same letter $F(\alpha)$. In the work of P. S. Novikov ([4]), by means of the function $F(\alpha)$ a certain class of functions is defined whose domain of definition is the set of all countable ordinals, and whose range is certain elements of one-dimensional Baire space. Each such function is denoted by $F(\alpha)$, and the value $F(\alpha)$ is called an image of the set $F(a)$. By a method differing only insignificantly from that set out in ([4]), for any set E of type proper CA one can construct in the two-dimensional Baire space (the direct product of the spaces x and y) a set θ with the following properties: (1) θ is a set of type A_r; (2) $\theta \cdot p_y$ is at most countable; (3) if the index of the sieve defining E coincides at the points y_1 and y_2, then $\operatorname{pr}_x \theta \cdot p_{y_1} = \operatorname{pr}_x \theta \cdot p_{y_2}$; (4) $\operatorname{pr}_y \theta \subseteq E$; (5) $\operatorname{pr}_x \theta \subseteq [1, 2]$; (6) $\operatorname{pr}_x \theta$ coincides with the set of all images in the interval $[1, 2]$. Denote $\operatorname{pr}_x \theta$ by D. Uniformize the set θ with respect to x by Kondo' s theorem. Denote the set obtained by θ'. Denote $\operatorname{pr}_y \theta'$ by W. The set W contains no perfect subset, for otherwise such a subset would be covered by a countable number of constituents of the set E, and one of these constituents would contain an uncountable part of W; the preimage of this part of the set W with respect to θ' would also be uncountable, which contradicts property (3) of the set θ.

We show that the set D coincides with all constructive real numbers x lying in the interval $[1, 2]$ (which, in particular, proves remark 2). Let an element x of the interval $[1, 2]$ be constructive and have the expansion (see $(^4)$) $1x_1 \dots x_{11} \dots$. An element with expansion $x_1 \dots x_n \dots$ is also constructive; therefore there is a value of the Gödel function at some countable ordinal α. Then $F(\alpha) \doteq x$. Indeed, the composition $F(\alpha)$ is $F(\beta_1) \dots$, where $F(\beta_1) \dots F(\beta_{11}) \dots$ are images of the elements of the set x. The elements of the set x are pairs of natural numbers $\langle x_1 1 \rangle \dots \langle x_n n \rangle \dots$, and their images coincide with their numbers $r(\langle x_1 1 \rangle) \dots r(\langle x_n n \rangle) \dots$. But the only element with composition $\{r(\langle x_1 1 \rangle) \dots r(\langle x_n n \rangle) \dots\}$ is the element with expansion $1x_1 \dots x_n \dots$. By property (6), $x \in D$. Let $x \in D$ and have expansion $1x_1 \dots x_n \dots$,

i.e., $F(\alpha) = x$. The composition of x is $\{r(\langle x_1 1 \rangle) \dots r(\langle x_n n \rangle) \dots\}$. Consequently, the set $F(\alpha)$ consists of pairs $\langle x_1 1 \rangle \dots \langle x_n n \rangle \dots$, i.e., the element $x_1 \dots x_n \dots$ is constructive, but then the element $1x_1 \dots x_n \dots$ is also constructive. From property (2) of the set θ it follows that W is countable, if the constructive real numbers are countable.

We pass to the proof of the second part of the theorem. Extend θ' from a perfect subset D of positive measure to the whole interval $[1, 2]$, preserving the type of the set θ'. In the same way as in the first part of the present proof, one can show that the extension of θ' is a nonmeasurable function. Consequently, one of its Lebesgue sets is a nonmeasurable B_2. The proof is complete.

Remark 1. One can construct a model ZF in which, simultaneously: a) there is an uncountable CA-set without a perfect kernel; b) there is a nonconstructive subset of the natural number sequence; c) the equality $2^{\aleph_0} = N_1$ is violated; d) the axiom of choice is refutable. This follows from [5] and from the fact that the theorem can also be proved without the axiom of choice.

Remark 2. From the proof of the theorem it follows that the set of constructive real numbers has type A_2. Another, more complicated proof of this fact is given in [6].

Remark 3. One can construct a model ZF in which the set of constructive real numbers is not a set of type A.

Remark 4. If the set of constructive real numbers is not of measure zero, then either the set of constructive real numbers is nonmeasurable (and has type A_2), or one can construct a nonmeasurable B_2.

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Note: Figure translations are in progress. See original paper for figures.