A GENERALIZATION OF LYAPUNOV' S FIRST STABILITY THEOREM FOR CERTAIN CLASSES OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 517.919.2:517.948

MATHEMATICS

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A GENERALIZATION OF LYAPUNOV' S FIRST STABILITY THEOREM FOR CERTAIN CLASSES OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

(Presented by Academician V. I. Smirnov on 20 IV 1967)

§ 1. We consider the differential equation

$$\dot{z} = Az + f(t, z) \tag{1}$$

with the initial condition

$$z(0) = z_0. \tag{2}$$

Here $z = z(t)$ is the sought continuous, continuously differentiable on the half-axis $[0, \infty)$, vector-valued function with values in an arbitrary complex Banach space Z; A is a linear operator with values in Z, with domain $D(A)$, dense in Z, and with spectrum in the left half-plane, generally speaking unbounded; $f(t, z)$ is a function defined and continuous on $[0, \infty) \times S$, where $S = [z;\ z \in Z;\ \|z\| < \rho]$, with values in Z, bounded with respect to t and, for each t, analytic in z, and its F-series has the form*

$$f(t, z) = \sum_{k=2}^{\infty} P_k^{(f)}(t)(z); \tag{3}$$

z_0 is a fixed element of Z.**

The main results of the present paper are stated in Theorems 1 and 2 and, briefly, are as follows:

1. The trivial solution of equation (1) is asymptotically stable.
2. The domain of asymptotic stability contains some sphere $\|z_0\| < \Lambda$, whose radius Λ is effectively determined.

3. The solution of equation (1) satisfying the initial condition (2) is an analytic function of the vector of initial conditions in the sphere $\|z_0\| < \Lambda$ and an analytic function of the solution of the corresponding linear problem

$$\dot{z} = Az; \qquad z(0) = z_0 \tag{4}$$

in the same sphere.

These results are a generalization of Lyapunov' s first theorem (^{1}) for the case when the operator A is bounded and the function $f(t, z)$ is nonanalytic; questions of stability of the trivial solution of equation (1) were investigated by M. G. Krein (^{2}). In the present paper the results of (^{3}) are essentially used.

§ 2. Following (^{3}), we shall say that a function of a complex argument x, holomorphic in a neighborhood of the point $x = 0$,

$$y(x) = \sum_{i=0}^{\infty} a_i x^i \qquad (a_i \geqslant 0)$$

* The notation of (^{4}) is used in the paper.
** The exact formulation of the problem is given in § 2.

majorizes the analytic function $\psi(z)$ with values in the complex B-space U, defined in a neighborhood of the point Θ_Z of the complex B-space Z,

$$\psi(z) = \sum_{i=0}^{\infty} P_i^{(\psi)}(z),$$

if $\|P_i^{(\psi)}\| \leq a_i$.

Let $\Omega(z, \varphi)$ be an analytic function of two variables in some neighborhood of the point (Θ_Z, Θ_Φ) of the space $Z \times \Phi$, with values in the space U (Z, Φ, U are complex Banach spaces):

$$\Omega(z, \varphi) = \sum_{i,j=0}^{\infty} P_{ij}^{(\Omega)}(z, \varphi). \tag{5}$$

We introduce the notation*

$$\left\|P_{ij}^{(\Omega)*}\right\| = \sup \left\|P_{ij}^{(\Omega)*}(z_1, z_2, \ldots, z_i, \varphi)\right\|, \qquad \|z_k\| = \|\varphi\| = 1 \quad (k = 1, 2, \ldots, i).$$

In what follows we shall assume that equation (1) satisfies conditions I or conditions II.

Conditions I.

Ia) The operator A belongs to the B-algebra of endomorphisms of the space Z and has spectrum $\sigma(A)$ lying in the left half-plane $\operatorname{Re}\lambda < 0$.

Ib) The power series

$$\bar{f}(x) = \sum_{k=2}^{\infty} \sup_{t\in[0,\infty)} \left\| P_k^{(f)*}(t) \right\| x^k \tag{6}$$

has a radius of convergence different from zero.

As is known from (^{2,4}), in this case the operator A is the infinitesimal generator of a uniformly continuous semigroup of linear bounded operators $T(t) = e^{At}$, and for any positive

$$\nu < \inf_{\lambda\in\sigma(A)} |\operatorname{Re}\lambda|$$

there exists a constant $N > 0$ such that

$$\|T(t)\| \leq N e^{-\nu t} \qquad (t \geq 0). \tag{7}$$

Conditions II.

IIa) A is an unbounded operator, a linear operator with domain of definition $D(A) \in Z$, with values in Z; the operator A is the infinitesimal generator of a strongly continuous semigroup $T(t)$ of class C_0, of negative type, of linear bounded transformations.

IIb) The function $f(t, z)$ does not depend on t.

As is known (^{4}), an operator A satisfying condition IIa) is closed and $D(A)$ is dense in Z. If $\omega_0 < 0$ is the type of the semigroup $T(t)$, then the spectrum of the operator A lies in the half-plane $\operatorname{Re}\lambda \leq \omega_0$, and for any positive $\nu < |\omega_0|$ there exists a constant $N > 0$ such that, for $\|T(t)\|$, inequality (7) holds.

Broad and important classes of operators satisfying condition IIa) are known (see, for example, (^{5,6})).

§ 3. Consider the equation

$$z = T(t)z_0 + \int_0^t T(t-s) f[s, z(s)]\, ds, \tag{8}$$

where $T(t)$ is a semigroup, strongly continuous for $t \geq 0$, of linear bounded operators satisfying inequality (7), and $f(t, z)$ satisfies condition Ib).

[*] $P_{ij}^{(\Omega)*}(z_1, z_2, \dots, z_i, \varphi)$ is the multilinear symmetric form that becomes $P_{ij}^{(\Omega)}(z, \varphi)$ when $z_1 = z_2 = \dots = z_i = z$. In paper (3) the following correction must be introduced: $\|P_{ij}^{(\Omega)}\|$ should everywhere be replaced by $\|P_{ij}^{(\Omega)*}\|$.

Introduce the notation: C is the Banach space of continuous bounded functions $\varphi(t)$ with values in Z, defined on $[0, \infty)$ ($\|\varphi\|_C = \sup_{t\in[0,\infty)} \|\varphi(t)\|_Z$); C_ν is the Banach space of continuous bounded functions $\varphi(t)$ with values in Z, defined on $[0, \infty)$, and such that there exists

$$\|\varphi\|_{C_\nu} = \sup_{t\in[0,\infty)} \|e^{\nu t}\varphi(t)\|_Z \quad (\nu > 0,\ \text{see § 2});$$

$$\tilde{C} = [u;\ u = T(t)z_0;\ z_0 \in Z;\ \|u\|_{\tilde{C}} = \|u\|_C]; \tag{9}$$

obviously, $\tilde{C}$ is a subspace of the space C;

$$\Lambda = \max_{x\geq 0} \frac{1}{N}\left[x - \frac{N}{\nu}\,\bar{f}(x)\right]; \tag{10}$$

$$S_1 = [z_0;\ z_0 \in Z;\ \|z_0\| < \Lambda]; \tag{11}$$

$$S_2 = [u;\ u \in \tilde{C};\ \|u\|_C < N\Lambda]. \tag{12}$$

Lemma 1. Equation (8) has a solution $z(z_0, t)$ possessing the following properties:

1) $z(z_0, t)$ is a function analytic in three senses: (α) from S_1 into C; (β) from S_1 into C_ν; (γ) from S_2 into C.
2) $z(z_0, t)$, analytic in the senses (α) and (β), is majorized by the function $x_1(\lambda)$, and in the sense (γ) by the function $x_2(\lambda)$, where $x_1(\lambda)$ and $x_2(\lambda)$ are analytic solutions of the equations

$$x = N\lambda + \frac{N}{\nu}\,\bar{f}(x), \tag{13}$$

$$x = \lambda + \frac{N}{\nu}\,\bar{f}(x), \tag{14}$$

respectively, satisfying the condition $x_1(0) = x_2(0) = 0$.

3) If X is the value of x at which the maximum in (10) is attained, then

$$\|z(z_0, t)\|_{C_\nu} \leq X.$$

§ 4. As is known (2), if the operator A satisfies condition Ia) of § 2, then equation (8) and equation (1) with initial condition (2) are equivalent. Taking into account the uniqueness (2) of the solution of the differential problem (1)–(2), we obtain from Lemma 1 the following assertion:

Theorem 1. Let equation (1) satisfy conditions I of § 2. Then:

1) The trivial solution of equation (1) is asymptotically stable, and the domain of asymptotic stability contains the sphere $\|z_0\| < \Lambda$, where Λ is determined from (10).
2) If X is the value of x at which the maximum in (10) is attained, then, for solutions of equation (1) with initial conditions (2) from the sphere $\|z_0\| < \Lambda$, the inequality

$$\|z(z_0, t)\|_Z \leq Xe^{-\nu t} \quad (t \geq 0)$$

holds.
3) The solution of the differential problem (1)–(2) is an analytic function of the vector of initial conditions z_0 from the sphere $\|z_0\| < \Lambda$ into the space C; it is also an analytic function from the sphere $\|T(t)z_0\|_C < N\Lambda$ into the space C of the solution of the corresponding linear problem (4).
4) The analytic functions of the preceding item of the theorem are majorized, respectively, by the functions $x_1(\lambda)$ and $x_2(\lambda)$ (see Lemma 1).

§ 5. **Lemma 2.** Let conditions II of § 2 hold; $u_0 \in S_2$ (see (12)) and $du_0/dt \in C$ with $\|du_0/dt\|_C = m$.

Write the solution of equation (8) in the form

$$z(z_0, t) = \sum_{k=1}^{\infty} P_k(t)(u_0) \tag{15}$$

(see item 1 of Lemma 1).

Then the terms $P_k(t)(u_0)$ of the series (15) have derivatives

$$\frac{d}{dt} P_k(t)(u_0) \in C, \qquad \left\| \frac{d}{dt} P_k(t)(u_0) \right\|_C \leq q_k \|u_0\|^k,$$

where q_k are the coefficients of the series

$$\sum_{k=1}^{\infty} q_k \lambda^k = \frac{m_0 \lambda}{1 - N v^{-1} f'_x\left[x_2(\lambda)\right]}. \tag{16}$$

Obviously, $q_k \geq 0$, and the series (16) converges for $|\lambda| < N\Lambda$. Here $m_0 = m/\|u_0\|$, and $x_2(\lambda)$ is as in Lemma 1.

Corollary. Suppose that the conditions II of § 2 are satisfied and that $z_0 \in D(A) \cap S_1$. Then:

1) The solution $z(z_0, t)$ of equation (8) has a continuous bounded derivative

$$\frac{d}{dt} z(z_0 t) \in C^*.$$

2) It is the unique solution of the differential problem (1)–(2). (Here the results of the works (7, 8) are used.)

Following the ideas of (9), we shall call a generalized solution of equation (1) with initial condition (2) the limit, as $n \to \infty$, in the space C, of a convergent sequence of solutions $\{z_n(t)\}$ of equation (1) satisfying the initial conditions z_{0n}, if, as $n \to \infty$, the sequence $\{z_{0n}\}$ converges to z_0.

It is easy to see that, for $z_0 \in S_1$, the solution of equation (8) is the unique generalized solution of the differential problem (1)–(2).

Theorem 2. Suppose that the conditions II of § 2 are satisfied. Then all the assertions of Theorem 1 are valid for the generalized solution of equation (1) with initial condition (2), except that in item 3 of Theorem 1, instead of “ ···the solution of the corresponding linear problem···” one should read “···the generalized solution of the corresponding linear problem···” .

§ 6. In conclusion we give a simple example.

Consider (9, 10) the equation of one-dimensional diffusion (the diffusion coefficient is assumed constant), proceeding simultaneously with an irreversible third-order chemical reaction in an absorbing medium:

$$\partial u/\partial t = \partial^2 u/\partial x^2 + ku^3; \quad u(t,0) = u(t,\pi) = 0; \quad u(0,x) = u_0(x). \tag{17}$$

Let Y be the set of real odd functions, defined on $[-\pi, \pi]$, which vanish at the endpoints of this interval and are such that the series of their Fourier coefficients converge absolutely. This set becomes a B-space if one introduces in it the norm

$$\|y\| = \sum_{k=1}^{\infty} |b_k^{(y)}|$$

($b_k^{(y)}$ are the coefficients of the Fourier series of the element y).

If one assumes that $u_0(x) \in Y$ and takes for Z the complex extension of Y, then it is easy to show that in problem (17) the conditions II of § 2 are satisfied with $v = 1,\ N = 1$. Therefore all the assertions of Theorem 2 are valid with

$$\Lambda = \frac{2}{3\sqrt{3}\,|k|}; \qquad X = \frac{1}{\sqrt{3}\,|k|}; \qquad \|z(z_0,t)\| < \frac{1}{\sqrt{3}\,|k|} e^{-t}.$$

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Received
13 IV 1967

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* If $z_0 \in D(A^n) \cap S_1$, then the solution $z(z_0, t)$ of equation (8) has $d^n z(z_0, t)/dt^n \in C$.

Note: Figure translations are in progress. See original paper for figures.

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