IRREGULAR SURFACES OF BOUNDED EXTERNAL CURVATURE AND INEQUALITIES OF ISOPERIMETRIC TYPE

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IRREGULAR SURFACES OF BOUNDED EXTERNAL CURVATURE AND INEQUALITIES OF ISOPERIMETRIC TYPE

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In the present note we consider surfaces with finite external positive curvature, which is defined, roughly speaking, as the measure of the set of locally supporting planes. The class of such surfaces includes regular and convex surfaces, polyhedra, and also surfaces studied earlier in ([1]-[6]). It is shown below that surfaces of the indicated type with rectifiable boundary have finite area, and under some additional assumptions (apparently connected only with the method of proof) have an intrinsic metric of bounded curvature in the sense of A. D. Aleksandrov.

Let M be a two-dimensional compact manifold (closed or with boundary); f a continuous mapping of M into n-dimensional Euclidean space E^n, $n \geq 2$; and F the Fréchet surface defined by the mapping f.

A domain $g \subset \operatorname{int} M$ will be called a ν-cap of the mapping f if in E^n there is a hyperplane P with normal ν such that $f(\partial g) \in P$ and $f(g)$ lies in the open half-space determined by the plane P and the vector ν.

For each open set $G \subset \operatorname{int} M$ define on the unit $(n-1)$-sphere S_{n-1} the multiplicity function $m_G(\nu)$, setting it equal to the greatest number of pairwise nonintersecting ν-caps of f contained in G (it is not excluded that $m_G(\nu) = +\infty$). The function $m_G(\nu)$ is measurable. Put

$$\mu_n^+(G) = \int_{S_{n-1}} m_G(\nu)\, d\sigma_\nu, \qquad \mu_n^+(A) = \inf_{G \supset A \cap \operatorname{int} M} \mu_n^+(G),$$

where A is an arbitrary subset of M.

Since $\mu_n^+(M)$ does not depend on the parametrization of the surface F, we shall use the notation $\mu_n^+(F)$, or simply μ_n^+, μ^+. If F is a regular surface in E^3, then $\mu_3^+(F) = \int K^+\, dS$, where K is the Gaussian curvature, $K^+ = \frac{1}{2}(|K| + K)$, and dS is the element of area of the surface F.

We assign the surface F to the class $\mathfrak{M}$ if F is a two-dimensional Fréchet surface in E^n satisfying the conditions: 1) $\mu_n^+(F) < \infty$; 2) the length of the boundary of the surface F is finite.

If the surfaces F_i converge to the surface F, then

$$\mu^+(F) \leq \lim_{\infty} \mu^+(F_i).$$

Therefore the closure of the class of regular surfaces with uniformly bounded positive curvatures and boundary lengths is contained in $\mathfrak{M}$.

Introduce the following notation: $S(F)$ is the Lebesgue area of the surface F; l is the length of the boundary of F; d is the diameter of the surface F, i.e. the diameter of the set $f(M) \subset E^n$; χ is the Euler characteristic of F. By the letter C with various indices we shall denote positive constants depending only on the dimension.

Theorem 1. *A surface F of class $\mathfrak{M}$ has finite area and satisfies the inequality*

$$S \leq B(\mu_n^+, \chi)(l + d)^{13/7} d^{1/7}. \tag{1}$$

As $B(\mu_n^+, \chi)$ one may take the function $\exp\{C_1(3+\mu_n^+ - \chi) \times \ln[C_2(3+\mu_n^+ - \chi)]\}$.

Theorem 2. *Let F be a nondegenerate* surface of class $\mathfrak{M}$ in three-dimensional Euclidean space E^3. Suppose that in a neighborhood of each of its interior points the surface F admits an explicit representation and that $\mu^+(u) < 2\pi$ for all $u \in \operatorname{int} M$. Then the metric of the space E^3 induces on F the intrinsic metric of a manifold of bounded curvature* ([7]), *and moreover $\omega^+(F) \leq \mu^+(F)$, where ω^+ is the intrinsic positive curvature of F.*

The proof of Theorem 1 is based on the following consideration. First an inequality of the form (1) is proved for a surface in E^2 (Theorems A, B). Let now F be a surface in E^n, and let F_p be the orthogonal projection of F onto the two-dimensional plane E_p^2, where p is a point of the Grassmann manifold G_n^2. From the equality

$$2\pi\omega_n^2 \mu_n^+(F) = V_{n-1} \int_{G_n^2} \mu_2^+(F_p)\, d\omega_p,$$

where ω_n^2 is the volume of G_n^2, V_{n-1} is the volume of S_{n-1}, and from the known estimate ([8]) of the area of a surface in terms of the areas of its projections onto pairwise orthogonal planes E_i^2, $i = 1, \dots, C_n^2$, it follows that there is a point $p \in G_n^2$ for which

$$C_4 S(F) \leq S(F_p), \qquad \mu_2^+(F_p) \leq C_5 \mu^+(F).$$

Therefore the assertion of Theorem 1 follows from estimates of the form (1) for F_p. Theorem 2 follows from the preceding theorem and Theorem 1 of the work ([6]).

A surface F in E^n will be called polyhedral if there exist a triangulation T of the manifold M and a parametrization f of the surface F such that on each triangle of the triangulation T the mapping f is a homeomorphism onto a plane rectilinear triangle in E^n.

Theorem A (Yu. D. Burago). *The area of a polyhedral surface satisfies an inequality of the form (1).*

If the surface F is closed, then the assertion of Theorem A follows from Theorem $1'$ of the note ([9]). In the case where F has nonempty boundary, the following plays an essential role.

Lemma 1. *If F is a simply connected polyhedral surface and $\mu^+(F) < \pi/2$, then $S \leq C_6 l^{13/7} d^{1/7}$.*

This assertion is obtained with the help of Theorem 2 of the note ([9]).

Let us outline the scheme of reducing Theorem A to Lemma 1. A surface will be called **even** if the length of the boundary of any closed simply connected domain $W \subset F$ satisfying the conditions $\omega^+(W) \geq \pi/24$, $\omega^+(\operatorname{int} W) \leq \pi/12$ is not less than $C_7(\mu^+ + 1)^{21} d$. The meaning of this definition is that, on the one hand, on an even surface the intrinsic curvature is distributed, in a certain sense, "sparsely," while, on the other hand, from an arbitrary polyhedral surface F one can pass to an even surface F' in such a way that the increase in boundary length and in the order of connectedness admit "good" estimates.

An estimate of the form (1) for an even surface F' is proved by induction on the Euler characteristic of the surface. Here the nodal case is an even surface F^* having the topological type of a cylinder. It can be shown that if the surface F^* did not satisfy an inequality of the form (1), then on it, or on its universal covering, there would be found a simply connected domain Q for which the following conditions are fulfilled:

* A surface F is called nondegenerate if the space of components $\Gamma(M, f)$ ([10]) is homeomorphic to M.

1) $S(Q)$ constitutes an essential part of $S(F^*)$; 2) $\mu^+(Q) < \pi/2$; 3) for Q the reverse isoperimetric inequality $S(Q) \geq C_8 l_Q^2$ holds, where l_Q is the length of the boundary of Q. Applying estimate (2) to the domain Q, we arrive at an inequality of the form (1).

Theorem B (S. Z. Shefel). *A surface F of class $\mathfrak{M}$ in E^2 can be approximated by a sequence of surfaces F_i satisfying the conditions: 1) the lengths of the boundaries F_i converge to the length of the boundary of F; 2) $\mu_2^+(F_i) \leq C_9\mu^+(F) + 2\pi|\chi(F)|$; 3) each surface F_i is, in the sense of Kerékjártó (see (10), p. 459), equivalent to a polyhedral surface.*

Remark 1. Theorem B generalizes the analogous theorem on saddle surfaces (5).

Remark 2. If the boundary of F consists of curves $L^1, \dots, L^r$ and polygonal lines $L_i^j \to L^j$ as $i \to \infty$, $j = 1, \dots, r$, then the sequence $\{F_i\}$ can be chosen so that

$$\partial F_i = \bigcup_{j=1}^{r} L_i^j.$$

Remark 3. If $\mu_2^+(F) < 2\pi$ and the surface F has the topological type of a disk or a cylinder, then condition 2) of the theorem may be replaced by the inequality $\mu_2^+(F_i) \leq \mu_2^+(F)$.

Remark 4. If the surface F is saddle, then the approximating surfaces F_i may also be chosen saddle.

The proof of Theorem B is based on Lemmas 2 and 3 given below.

The mapping $\dot{f}$ admits a decomposition $f = m \circ \gamma$

$$M \xrightarrow{m} \Gamma(M, f) \xrightarrow{\gamma} E^2,$$

where $\Gamma(M, f)$ is the space of components (10). We shall call a surface F (k, p)-convex if there exists a cellular decomposition of the space $\Gamma(M, f)$ such that on each two-dimensional cell γ is a homeomorphism onto a convex domain in E^2, the number of two-dimensional cells not exceeding k, and the total length of the curves defined by the mapping γ on the one-dimensional skeleton not exceeding p.

Introduce in E^2 the norm $\|x\| = \max(|x^1|, |x^2|)$, where x^1, x^2 are the coordinates of the vector x in some fixed coordinate system.

Denote by $H = H(F, L, \varepsilon, k, p)$ the class of (k, p)-convex surfaces Φ having common boundary L and at distance from F, in the Fréchet metric induced by the norm introduced above, not more than ε.

Lemma 2. *The space H is compact.*

Let φ, ψ be mappings of M into E^2. A v-hill g of the mapping φ will be called **special** (with respect to ψ) if g is homeomorphic to a disk or an annulus and contains no v-hill of the mapping ψ.

Lemma 3. *Let $\|\varphi - \psi\| \leq \varepsilon$ and let ψ have no hills defined by vectors parallel to the coordinate axes. If g is a special v-hill of the mapping φ, and π is a line containing $\varphi(\partial g)$, then there exists a mapping φ^*, coinciding with φ on $M \setminus g$, monotone on g, and taking g into the line π.*

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Note: Figure translations are in progress. See original paper for figures.