BOUNDARY-VALUE PROBLEMS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS IN A HILBERT SPACE WITH A VARIABLE SELF-ADJOINT OPERATOR

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 517.948:513.88

MATHEMATICS

G. I. LAPTEV

BOUNDARY-VALUE PROBLEMS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS IN A HILBERT SPACE WITH A VARIABLE SELF-ADJOINT OPERATOR

(Presented by Academician I. N. Vekua on 10 V 1967)

In this note the main results of the works (^{1},^{2}) are carried over to differential equations with a variable self-adjoint operator in a Hilbert space. In constructing the Green function of one special boundary-value problem, some ideas of Giraud, set forth in the book (^{3}) (Chap. III, Sec. 20), are developed.

Let the differential equation be given on the interval $[0, T]$

$$d^2u/dt^2 = A(t)u - f(t) \qquad (0 \leq t \leq T), \tag{1}$$

where $u(t)$ is the unknown function with values in a certain Hilbert space H; $A(t)$, for each $t \in [0, T]$, is an unbounded linear self-adjoint operator with domain of definition dense in H; $f(t)$ is a given continuous function with values in H.

It is also assumed that the operators $A(t)$ are uniformly positive:

$$(A(t)x, x) \geq \delta(x, x) \qquad (x \in D(A) \text{ and } \delta > 0). \tag{2}$$

For equation (1) we consider a boundary-value problem, i.e., the problem of finding a solution $u(t)$ satisfying a system of linear boundary conditions of the form

$$L_i(u) = \alpha_{i1}u(0) + \alpha_{i2}u'(0) + \beta_{i1}u(T) + \beta_{i2}u'(T) = f_i \qquad (i = 1, 2), \tag{3}$$

where α_{ij}, β_{ij} $(i, j = 1, 2)$ are complex numbers; f_i are certain elements of the space H.

For the boundary-value problem (1)–(3) we retain the definition of a solution $u(t)$ given in (^{1}).

As usual, by $L_2([0,T],H)$ we denote the Hilbert space of Bochner-integrable functions with values in H, where the scalar product is defined by the formula

$$(u,v)_{L_2} = \int_0^T (u(t),v(t))\,dt.$$

Lemma. *Let the solution $u(t)$ of the differential equation (1) be such that* $\operatorname{Re}(u'',u)_{L_2} \le 0$. *Then* $\|u\|_{L_2} \le \delta^{-1}\|f\|_{L_2}$.

For the proof it suffices to multiply equation (1) scalarly in L_2 by $u(t)$ and apply inequality (2).

Integrating by parts, we find that

$$(u'',u)_{L_2} = (u',u)\big|_0^T - \|u'\|^2_{L_2}.$$

Consequently, the condition of the lemma will be fulfilled if

$$\operatorname{Re}(u',u)\big|_0^T \le 0.$$

This last inequality is satisfied by the homogeneous boundary conditions of many problems, for example, the first and second problems, and the problem with periodic conditions (see, in particular, (4)).

The lemma ensures the uniqueness of the solution of all such problems.

In what follows, homogeneous boundary conditions of the form

$$u'(0)=0, \qquad u'(T)=0. \tag{4}$$

will play a special role.

It is known how important the Green function is in the theory of boundary-value problems. In this connection we introduce a definition.

An operator function $G(t,\tau)$ is called the Green function of the problem (1)–(3) if, for every function $f(t)$ satisfying the Hölder condition, the integral

$$g(t) = \int_0^T G(t,s)f(s)\,ds \tag{5}$$

gives a particular solution of equation (1) which satisfies the homogeneous boundary conditions $L_1(g)=L_2(g)=0$.

We shall now prove the existence of the Green function for the special boundary-value problem (1)–(4) under the following assumptions:

$\alpha)$

the domain of definition D of the operator $A(t)$ does not depend on t;

$\beta)$

for any $t, \tau, s \in [0, T]$ and some $\varepsilon > 0$ the inequality

$$\big\|[A(t) - A(\tau)]A^{-1}(s)\big\| \leq C_0|t - \tau|^{\varepsilon}$$

holds.

In the proof we shall use the following guiding considerations. Let $F(t, \tau)$ be the Green function for the equation $v'' = A(\tau)v - g(t)$, with fixed τ, under certain boundary conditions $L_1(v) = L_2(v) = 0$. If $G(t, \tau)$ is the Green function for equation (1) under the same boundary conditions, then the difference $W(t, \tau) = G(t, \tau) - F(t, \tau)$ will be a solution of the problem

$$d^2W/dt^2 = A(t)W - [A(\tau) - A(t)]F; \qquad L_1(W)_t = L_2(W)_t = 0,$$

and therefore can be represented in the form of the integral (5). In other words, $G(t, \tau)$ serves as a solution of the integral equation (in the variable t)

$$G(t, \tau) = F(t, \tau) + \int_0^T G(t, s)[A(\tau) - A(s)]F(s, \tau)\, ds. \tag{6}$$

Equation (6) will serve as the basis for constructing the function $G(t, \tau)$.

Theorem 1. *If the positive number k is sufficiently large and conditions $\alpha)$ and $\beta)$ are fulfilled, then under the boundary conditions (4) there exists a Green function $G_k(t, \tau)$ for the equation $u'' = [A(t) + k^2]u - f(t)$.*

The proof is based on the fact that, under the conditions of the theorem, the kernel of equation (6) admits the estimate

$$\|[A(\tau) - A(s)]F(s, \tau)\| \leq C|s - \tau|^{\varepsilon - 1}e^{-k|s-\tau|}.$$

Here $U(t)$ is the semigroup generated by the operator $-[k^2 + A(\tau)]^{1/2}$, and

$$\begin{aligned} F(t, \tau) = \frac{1}{2}[k^2 + A(\tau)]^{-1/2}\{&U(|t - \tau|) + [I - U(2T)]^{-1}[U(t + \tau) \\ &+ U(2T + t - \tau) + U(2T - t + \tau) + U(2T - t - \tau)]\}. \end{aligned}$$

The resulting Green function has the following symmetry property:

$$G_k(t,\tau) = G_k^*(\tau, t). \tag{7}$$

Theorem 2. *If conditions α) and β) are fulfilled, then there exists a Green function $G(t,\tau)$ of the boundary-value problem (1)–(4).*

Proof. The desired function satisfies the resolvent equation

$$G(t,\tau)x = G_k(t,\tau)x + k^2 \int_0^T G_k(t,s)G(s,\tau)x\,ds \qquad (x \in H). \tag{8}$$

The kernel $G_k(t,s)$, by virtue of property (7), defines a self-adjoint bounded operator in the Hilbert space $L_2([0,T];H)$. Thus the point k^2 may be either an eigenvalue (point spectrum), or a point of the continuous spectrum, or else a point of the resolvent set.

Suppose that k^2 is an eigenvalue, i.e., there exists a function $v(t)$ such that

$$v(t) = k^2 \int_0^T G_k(t,s)v(s)\,ds.$$

But then

$$\frac{d^2v}{dt^2} - A(t)v = k^2 \left[\int_0^T k^2 G_k(t,s)v(s)\,ds - v(t)\right] = k^2[v(t) - v(t)] = 0,$$

i.e., $v(t)$ satisfies the homogeneous differential equation (1) and the homogeneous boundary conditions (4). As was already noted, this problem has a unique solution, and therefore $v(t) \equiv 0$, and k^2 cannot be an eigenvalue.

Suppose that k^2 belongs to the continuous spectrum. This means that there exists a sequence of functions $v_n(t)$ on the unit sphere of the space $L_2([0,T];H)$ for which

$$v_n(t) - k^2 \int_0^T G_k(t,s)v_n(s)\,ds = \varphi_n(t), \tag{9}$$

where $\|\varphi_n\|_{L_2} \to 0$ as $n \to \infty$. Denote the integral in (9) by $z_n(t)$ and once again apply the operator

$$\frac{d^2z_n}{dt^2} - A(t)z_n = \int_0^T k^2 G_k(t,s)v_n(s)\,ds - v_n(t) = -\varphi_n(t).$$

Thus the function $z_n(t)$ satisfies the boundary conditions (4) and the equation $z_n'' = A(t)z_n - \varphi_n(t)$.

The condition of the lemma is fulfilled; therefore the inequality $\|z_n\|_{L_2} \leq \delta^{-1}\|\varphi_n\|_{L_2}$ must hold, whence it follows that $\|z_n\|_{L_2} \to 0$ as $n \to \infty$. But $v_n(t) = \varphi_n(t) + k^2 z_n(t)$, where k is a fixed number; consequently, $\|v_n\|_{L_2} \to 0$ as $n \to \infty$, which contradicts the choice of the functions $v_n(t)$.

Thus the number k^2 is a resolvent point of the kernel $G_k(t, s)$. Equation (8) is therefore solvable, and a direct verification shows that the function $G(t, \tau)$ determined by it is the Green' s function of problem (1)–(4). The theorem is proved.

The constructed function $G(t, \tau)$ has the following property: the functions $U_1(t) = G(t, 0)$ and $U_2(t) = G(t, T)$ are particular linearly independent solutions of the homogeneous differential equation (1). On this basis one can construct the theory of boundary-value problems (1)–(3). Thus, the following holds.

Theorem 3. *Let the function $f(t)$ satisfy the Hölder condition. If $u(t)$ is a solution of equation (1), then it can be represented in the form*

$$u(t) = U_1(t)g_1 + U_2(t)g_2 + \int_0^T G(t, s)f(s)\,ds, \tag{10}$$

where g_1, g_2 are certain elements of the space H and $G(t, \tau)$ is the Green' s function of problem (1)–(4).

Conversely, if $f(t)$ satisfies the Hölder condition, then for any elements $g_1, g_2 \in H$ formula (10) defines a solution of equation (1).

Substitution of (10) into the boundary conditions (3) gives a system of equations for g_1 and g_2 (by $g(t)$ we denote the integral in (10)):

$$L_i(U_1)g_1 + L_i(U_2)g_2 = f_i - L_i(g) \qquad (i = 1, 2). \tag{11}$$

It is convenient to regard the resulting system as a single equation in the direct product $H \times H$. Let the operator matrix of this system be denoted by $\mathbf{D}$. If there exists a bounded operator $\mathbf{D}^{-1}$, then the system (11), and with it the original boundary-value problem (1)–(3), is thereby uniquely solvable. Thus the question of existence and uniqueness of the solution of problem (1)–(3) is reduced to the question of invertibility of the operator $\mathbf{D}$.

Here we note the following assertions.

Theorem 4. *The operator $\mathbf{D}^{-1}$ (possibly unbounded) exists if and only if the homogeneous boundary-value problem corresponding to problem (1)–(3) has only the zero solution.*

If the boundary conditions are regular, then the operator $\mathbf{D}$ can be represented in the form $C(I-\mathbf{R})$, $A^{-1/2}C(I-\mathbf{R})$, and $A^{-1}C(I-\mathbf{R})$, where I is the identity operator in $H \times H$ and C is bounded and has a bounded inverse. In particular, if $A^{-1}(t)$ is completely continuous, then the operator $\mathbf{R}$ is also completely continuous. If unity is not an eigenvalue of the operator $\mathbf{R}$, then the Green' s function exists and is a bounded function. For nonregular boundary conditions this is not so. Hence

Theorem 5. *Correct formulations of boundary-value problems are possible only under regular boundary conditions.*

Here, as usual, a boundary-value problem is called correct if its solution exists, is unique, and depends continuously on the initial data.

The most complete analogy with the scalar case is obtained for equations in which the operators $A^{-1}(t)$ are completely continuous.

Theorem 6. *Let the operators $A^{-1}(t)$ be completely continuous and let the boundary conditions be regular. In order that the boundary-value problem (1)–(3) be correct, it is necessary and sufficient that the corresponding homogeneous boundary-value problem have only the zero solution.*

The author expresses gratitude to S. G. Krein for discussion of the paper.

Latvian State University
named after P. Stuchka

Received
6 V 1967

REFERENCES

1. S. G. Krein, G. I. Laptev, *Differents. Uravn.*, 2, No. 3 (1966).
2. S. G. Krein, G. I. Laptev, *Differents. Uravn.*, 2, No. 7 (1966).
3. K. Miranda, *Equations with Partial Derivatives of Elliptic Type*, Moscow, 1957.
4. V. A. Trenogin, *DAN*, 170, No. 5 (1966).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.