ON THE SPECTRAL PROPERTIES OF INDECOMPOSABLE OPERATORS

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Abstract

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MATHEMATICS

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ON THE SPECTRAL PROPERTIES OF INDECOMPOSABLE OPERATORS

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Let E be a semi-ordered Banach space with cone K ([1,2]); let $A(E \to E)$ be a linear positive operator; $r(A)$ its spectral radius. Throughout the paper it is assumed that K is a normal reproducing cone. By $\nu(A)$, $\sigma(A)$, $\sigma_c(A)$ we denote respectively the resolvent set, the spectrum, and the continuous spectrum of the operator A; $r(A)$ is its spectral radius. The paper studies the spectral properties of an indecomposable operator.

1. In this section a variational characterization of the spectral radius of an indecomposable operator will be given. By P, Q we denote respectively the collection of all nonzero vectors $u, v \in K$ for which, for some $\mu, \lambda \geq 0$, the inequalities $Au \geq \mu u$, $Av \leq \lambda v$ hold. We first present one assertion valid for the case of a solid minihedral cone K.

Theorem 1. *Let K be solid and minihedral, and let the operator A be indecomposable. Suppose there exists a sequence of real numbers λ_n such that $\lambda_n \leq r(A)$, $\lambda_n \in \nu(A) \cup \sigma_c(A)$, and $\lim_{n\to\infty} \lambda_n = r(A)$. Then*

$$r(A) = \sup_{u\in P} \sup\{\mu : \ Au \geq \mu u\} = \inf_{v\in Q} \inf\{\lambda : \ Av \leq \lambda v\}. \tag{1}$$

The analogues of Theorem 1 for nonsolid cones are the following two assertions.

Theorem 2. *Let K be a minihedral cone, and let A be a linear u_0-positive operator ([2]). Suppose there exists a sequence of real numbers $\lambda_n < r(A)$ such that $\lambda_n \in \sigma(A)$ and $\lim_{n\to\infty} \lambda_n = r(A)$. Then relation (1) holds.*

Theorem 3. *Let K be a minihedral cone, and let the operator A be indecomposable, quasifully continuous, and $r(A) > 0$. Then relation (1) holds.*

We note that the assumption of indecomposability of the operator A in Theorems 1 and 3, and also the assumption of u_0-positivity of the operator A in Theorem 2, are essential—without these assumptions the assertions of Theorems 1-3 lose their force.

We give one corollary of Theorem 1. Let a nonlinear, completely continuous monotone operator F act in E and leave invariant the solid cone K. Suppose there exist linear positive operators A and B, $\rho > 0$, and elements $g, u_0 \in K$ such that $Fx \geq Ax$ for all $x \in K$, $\|x\| \leq \rho$, and $Fx \leq Bx + g$ for all $x \geq u_0$. Finally, suppose that the operator A satisfies the hypotheses of Theorem 1 and $r(A) > 1$, while $r(B) < 1$. Then the operator F has in K at least one fixed point $x^* \neq \theta$. A fixed point x^* of the operator F can be obtained by the method of successive approximations $u_n = Fu_{n-1}$, $v_n = Fv_{n-1}$, with a suitable choice of initial approximations $u_0, v_0 \in K$. Moreover,

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq x^* \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

2. In this section we consider the question of when the inequality $r(A) < r(B)$ follows from the inequality $\theta \leq A \leq B$, where A, B are linear operators. Below we put $C = B - A$.

Theorem 4. *Let K be solid, and let the operator C be indecomposable. Then every positive eigenvalue of the operator A (if it exists) is less than $r(B)$.*

Corollary. *Let the hypotheses of Theorem 4 be satisfied, and suppose that A is a quasibounded continuous operator. Then $r(A) < r(B)$.*

We note that in the case of a minihedral cone, under the hypotheses of Theorem 4 it can be proved that the absolute values of all eigenvalues of the operator A are less than $r(B)$.

Theorem 5. *Let the hypotheses of Theorem 4 be satisfied, and let $r(B)$ be a positive eigenvalue of the operator B. Then $r(A) < r(B)$.*

Let us make one more remark concerning Theorems 4 and 5. If the operators A, B are quasiboundedly continuous, then the assertions of these theorems remain valid also for nonsolid cones. The assumption that K is solid can be omitted in the hypotheses of these theorems also in the case when one of the operators is u_0-bounded from above.

3. In this section we give one assertion supplementing assertions of the "incompatible inequalities" type proved in ([2]), and indicate one of its applications. We shall write $x \ll y$ if $y - x \in K$.

Theorem 6. *Let a linear positive operator A be indecomposable. Suppose that one of the following three conditions is satisfied: a) the cone K is solid; b) the operator A is u_0-bounded from above; c) the operator A is quasiboundedly continuous.*

Then: 1) for no nonzero $u, v \in K$ and real $\lambda, \varepsilon > 0$ are the relations $Av \ll \lambda v$, $Au \gg (\lambda+\varepsilon)u$ compatible; 2) from the inequalities $Av_0 \ll \lambda_0 v_0$, $Au_0 \gg \lambda_0 u_0$ it follows that $\lambda_0 = r(A)$, $v_0 = cv_0$ for some real c, and $Au_0 = r(A)u_0$; 3) for $\lambda < r(A)$ and arbitrary $v \in K$, $v \neq 0$, the relation $Au \ll \lambda v$ holds.

From Theorem 6 and Theorem 4.12 of the monograph ([2]) the following assertion follows:

Theorem 7. *Let a nonlinear positive operator F be completely continuous and satisfy the following conditions: there exist $0 < r < R$ and such linear positive operators A_1 and A_2 that $A_1 x \ll Fx$ ($x \in K$, $0 < \|x\| < r$) and $Fx \ll A_2 x$ ($x \in K$, $\|x\| \geq R$). Suppose that one of conditions a)-c) of Theorem 6 is satisfied, A_1 is indecomposable and $r(A_1) > 1$, $r(A_2) \leq 1$. Then the operator F has in K at least one fixed point x^*, and moreover $r \leq \|x^*\| \leq R$.*

Under the hypotheses of Theorem 7 the operator F is a compression operator of the cone ([2]). Starting from Theorem 6 and Theorem 4.14 ([2]), it is easy to formulate an assertion analogous to Theorem 7 concerning an operator F that is an expansion of the cone.

4. Let the cone K be solid, and let A be an indecomposable completely continuous operator. Then $r(A) > 0$, and the number $r(A)$ is a positive and simple eigenvalue of the operator A ([5]). Denote by $x^*(\|x^*\| = 1)$ the eigenvector corresponding to $r(A)$, lying in K. In this section a method is indicated for approximately finding $r(A)$ and x^*.

Let u_0 be an arbitrary interior element of K. It is easy to see that for every n, for some $\alpha, \beta > 0$, the inequality $\beta u_0 \ll A^n u_0 \ll \alpha u_0$ holds. Denote by β_n^n (α_n^n) the exact upper (lower) bound of all such numbers β (α) for which the last inequality holds. Obviously,

$$\beta_n^n u_0 \ll A^n u_0 \ll \alpha_n^n u_0.$$

It turns out ([6]) that $\beta_n \leq r(A) \leq \alpha_n$, and moreover

$$\lim \beta_n = \lim \alpha_n = r(A).$$

Put

$$u_n = \sum_{k=1}^{n} \beta_n^{n-k} A^{k-1} u_0, \qquad v_n = \sum_{k=1}^{n} \alpha_n^{n-k} A^{k-1} u_0.$$

Then, as is not difficult to see,

$$\beta_{nu} n \ll A u_n \quad \text{and} \quad A v_n \ll \alpha_{nv} n,$$

whence

$$\beta_n A \frac{u_n}{\|Au_n\|} \ll A\left(A \frac{u_n}{\|Au_n\|}\right), \qquad A\left(A \frac{v_n}{\|Av_n\|}\right) \ll \alpha_n A \frac{v_n}{\|Av_n\|}.$$

Theorem 8. *Let the cone K be solid, and let A be an indecomposable completely continuous operator. Then the sequences $Au_n/\|Au_n\|$, $Av_n/\|Av_n\|$ converge to the unique normalized positive eigenvector x^* of the operator A: $Ax^* = r(A)x^*$.*

From the relations defining u_n, v_n, it follows that Theorem 8 is close to the ergodic theorems ([3]). We note that the assertion of Theorem 8 will also hold in the case where, instead of the assumption of complete continuity of the operator A, one makes the following assumption: the space E is weakly complete, the unit ball in E is weakly compact, and the cone K admits plastering ([2]). Similarly, the assumption that the cone is solid may be replaced by the assumption that the operator A is u_0-positive.

5. We present assertions supplementing the results of M. G. Krein ([1]) and I. A. Bakhtin ([7]) on commutative families of linear positive operators.

Theorem 9. *Let K be a solid cone. Then, whatever commutative family $\Gamma = \{A\}$ of linear positive operators A is given, there is always a positive functional $\varphi_0 \in E^*$, $\varphi_0 \neq \theta$, such that $A^*\varphi_0 = r(A)\varphi_0$ for all $A \in \Gamma$.*

An analogous assertion will hold if the assumption that the cone K is solid is replaced by the following condition: there exists an element $u_0 \in K$ such that for every $A \in \Gamma$, for some $a = a(x, A)$, the inequality $Ax \leq au_0$ is satisfied.

We give one more assertion about a commutative family $\Gamma = \{A\}$ of linear positive operators.

Theorem 10. *Let K be a solid cone and let at least one of the operators $A \in \Gamma$ be quasi-completely continuous and indecomposable. Then in E one can introduce a new equivalent norm $\| \ \|_1$ such that for every $A \in \Gamma$ the equality $\|A\|_1 = r(A)$ holds.*

It turns out that, under some additional conditions (for example, in the case of a reflexive space E), the new equivalent norm can be introduced so that it is strictly monotone: from $0 \leq x < y$ it follows that $\|x\|_1 < \|y\|_1$.

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Note: Figure translations are in progress. See original paper for figures.