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ON FRACTIONAL INTEGRATION BY PARTS

MATHEMATICS

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Abstract

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MATHEMATICS

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ON FRACTIONAL INTEGRATION BY PARTS

(Presented by Academician I. N. Vekua on March 1, 1968)

Consider the problem

$$v'(t) + Av(t) = f(t) \quad (0 \le t \le 2\pi), \qquad v(0) = v_0 \tag{1}$$

in a Hilbert space H with a positive definite self-adjoint operator A. By its solution we shall mean a function $v(t)$ satisfying (1) such that the functions $v'(t)$ and $Av(t)$ are continuous. If a solution of problem (1) exists, then it is given by the formula

$$v(t) = \exp\{-tA\}v_0 + \int_0^t \exp\{-(t-s)A\}f(s)\,ds = \exp\{-tA\}v_0 + Qf(t), \tag{2}$$

where $\exp\{-tA\}$ is the semigroup generated by the operator A. Necessary conditions for the existence of a solution of problem (1) are the continuity of the function $f(t)$ and the membership of v_0 in the domain $D(A)$ of the operator A. Are these conditions sufficient for the existence of a solution of problem (1)? If $v_0 \in D(A)$, then problem (1) has a solution in the case when the problem

$$v'(t) + Av(t) = f(t) \quad (0 \le t \le 2\pi), \qquad v(0) = 0. \tag{3}$$

has a solution.

In the present paper, the solvability of problem (3) is proved under the assumption that $f(t)$ belongs to the abstract space $W_2^{1/2}$ of L. N. Slobodetskii. We note that $f(t) \in W_2^{1/2}$ need not satisfy any Hölder condition. This result is based on the method of estimating integrals of the form

$$I_i = \int_\Omega u\,\frac{\partial v}{\partial x_i}\,dx, \qquad i = 1, \dots, n, \tag{4}$$

of products of smooth scalar functions $u(x)$ and $v(x)$ (or functions with values in a Hilbert space) defined in an n-dimensional domain Ω.

1. Theorem 1. Let the boundary S of the domain Ω be continuously differentiable once. Then the inequality

$$|I_i| \leq M(\alpha)\|u\|_{W_2^\alpha(\Omega)}\|v\|_{W_2^{1-\alpha}(\Omega)} \tag{5}$$

holds for any α from $[0, 1/2)$, and the inequality

$$|I_i| \leq M(1/2)(\|u\|_{W_2^{1/2}(\Omega)} + \|u\|_{L_2(S)})(\|v\|_{W_2^{1/2}(\Omega)} + \|v\|_{L_2(S)}). \tag{6}$$

We shall prove, for the one-dimensional integral

$$I = \int_0^{2\pi} u(x)v'(x)\,dx,$$

the one-dimensional analogues of these inequalities,

$$|I| \leq M(\alpha)\|u\|_{W_2^\alpha}\|v\|_{W_2^{1-\alpha}} \qquad (0 \leq \alpha < 1/2), \tag{7}$$

$$|I| \leq M(1/2)(\|u\|_{W_2^{1/2}} + |u(0)| + |u(2\pi)|)(\|v\|_{W_2^{1/2}} + |v(0)| + |v(2\pi)|). \tag{8}$$

Consider inequality (7). Let the function $u(x)$ be periodic ($u(0) = u(2\pi)$). Setting $v(x) = v_1(x) + v_2(x)$, where $v_2(x) = (t/2\pi)[v(2\pi) - v(0)]$, we obtain

$$I = I_1 + I_2 = \int_0^{2\pi} u(x)v_1'(x)\,dx + \frac{1}{2\pi}\int_0^{2\pi} u(x)\,dx\,[v(2\pi) - v(0)].$$

Let

$$u(x) = \sum_{|n|=0}^{+\infty} a_n e^{inx}, \qquad v_1(x) = \sum_{|n|=0}^{+\infty} b_n e^{inx}.$$

Then

$$|I_1| \leq 2\pi \sum |n|\,|a_n|\,|b_n| \leq 2\pi \left(\sum |n|^{2\alpha}|a_n|^2\right)^{1/2} \left(\sum |n|^{2-2\alpha}|b_n|^2\right)^{1/2}$$

But

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$$2\pi\sum(1+|n|^{2\alpha})|a_n|^2=\|u\|^2_{W_2^\alpha},\qquad 2\pi\sum(1+|n|^{2-2\alpha})|b_n|^2=\|v_1\|^2_{W_2^{1-\alpha}},$$

since the functions $u(x)$ and $v_1(x)$ are periodic. Therefore

$$|I_1|\le\|u\|_{W_2^\alpha}\|v_1\|_{W_2^{1-\alpha}}.$$

Further, for $\alpha>0$,

$$\|v_1\|_{W_2^{1-\alpha}}\le\|v\|_{W_2^{1-\alpha}}+\|v_2\|_{W_2^{1-\alpha}}\le\|v\|_{W_2^{1-\alpha}}+C_1(\alpha)|v(2\pi)-v(0)|.$$

By the embedding theorems,

$$|v(x)|\le C_2(\alpha)\|v\|_{W_2^{1-\alpha}}$$

for $\alpha<1/2$. Finally,

$$\left|(1/2\pi)\int_0^{2\pi}u(x)\,dx\right|\le C_3\|u\|_{W_2^\alpha}.$$

Hence inequality (7) follows for a periodic smooth function $u(x)$. However, for $\alpha<1/2$ a smooth function $u(x)$ can be approximated in the metric W_2^α by smooth periodic functions. Obviously, it is enough to prove this assertion for the linear function $(t/\pi)[u(2\pi)-u(0)]$, which can be approximated in the metric W_2^α, for $\alpha<1/2$, by its partial Fourier sums (these partial sums are periodic functions, and their Fourier coefficients decrease as c/n).

To prove inequality (8), the functions $u(x)$ and $v(x)$ must be reduced to periodic ones by the method indicated above.

2. The closure of the set K of all smooth functions $f(t)$, defined on $[0,2\pi]$ with values in H, in the norm

$$\|f\|_{L_2}=\left(\int_0^{2\pi}\|f\|^2_H(t)\,dt\right)^{1/2}$$

forms the space $L_2=L_2([0,2\pi],H)$. The closure of the set K in the norm

$$\|f\|_{W_2^1}:=(\|f\|^2_{L_2}+\|f'\|^2_{L_2})^{1/2}$$

forms the space W_2^1. The closure of the set K in the norm

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$$\|f\|_{W_2^\alpha} = \left(\|f\|_{L_2}^2 + \int_0^{2\pi} \int_0^{2\pi} \|f(t) - f(s)\|_H^2 |t-s|^{-1-2\alpha}\, dt\, ds \right)^{1/2} \qquad (0 < \alpha < 1)$$

forms the space W_2^α. The spaces W_2^l are defined analogously for any (integer or fractional) l.

If in the preceding constructions one considers, instead of K, its subset $\dot{K}$, consisting of functions $f(t)$ for which $f(0) = 0$, then we obtain the spaces $\dot{L}_2, \dot{W}_2^1, \dot{W}_2^\alpha$. Obviously, $\dot{L}_2 = L_2$. Just as in item 1, it is shown that $\dot{W}_2^\alpha = W_2^\alpha$ for $\alpha < 1/2$.

It is known (see, for example, (1)) that the operators AQ and $\dfrac{d}{dt}Q$ act boundedly in the spaces L_2 and $\dot{W}_2^1$. Hence it follows that

Theorem 2. The operators AQ and $\dfrac{d}{dt}Q$ act boundedly in the spaces $\dot{W}_2^\alpha$.

For the proof one may use the method applied in (2), p. 158, item 7, for spaces of scalar functions. In doing so, instead of the space $L_2((-\infty, +\infty), H)$ one must take its subspace L_2, consisting of odd functions, and take into account that the Fourier transform is a unitary operator in L_2, that in it there acts the operator J of multiplication by the function $(1+|t|^{\frac{1}{2}})^{1/2}$, and that it is a positive definite self-adjoint operator. Next, the domains of definition of the fractional powers J^α of the operator J form the spaces $\overset{\circ}{W}{}_2^\alpha$. Finally, the traces of functions from these spaces on the segment $[0, 2\pi]$ form the spaces $\overset{\circ}{W}{}_2^\alpha$ introduced above. Therefore between the spaces L_2 and $\overset{\circ}{W}{}_2^1$ one can stretch a Hilbert scale H_α so that the norms in H_α and in $\overset{\circ}{W}{}_2^\alpha$ are equivalent. Applying the interpolation theorem, we obtain the assertion of Theorem 2.

From Theorem 2 and the interpolation theorem it follows that

Theorem 3. *The operator $A^\alpha Q$ acts boundedly from L_2 to $W_2^{1-\alpha}$ and from $\overset{\circ}{W}{}_2^\beta$ to $\overset{\circ}{W}{}_2^{\beta+1-\alpha}$.*

3. Denote by $C^\alpha = C^\alpha([0, 2\pi], H)$ $(0 \le \alpha \le 1)$ the closure of the set K in the norm

$$\|f\|_{C^\alpha} = \max_{0 \le t \le 2\pi} \|f(t)\|_H + \sup_{0 \le t,\ s \le 2\pi} |t-s|^{-\alpha} \|f(t) - f(s)\|_H.$$

Theorem 4. *For any α from $[0, 1]$, $T > 0$, v_0 from H, the function $v(t) = \exp\{-tA\}A^{-\alpha}v_0$ belongs to $C^\alpha([0, T], H) \cap W_2^\beta([0, T], H)$ for $\beta = (1+\alpha)/2$, and the inequality*

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$$\|v(t)\|_{C^\alpha \cap W_2^\beta} \le M_\alpha \|v_0\|_H, \tag{9}$$

holds, where M_α depends neither on T nor on v_0. Here $\|\cdot\|_{C^\alpha \cap W_2^\beta} = \|\cdot\|_{C^\alpha} + \|\cdot\|_{W_2^\beta}$.

Proof. If $v_0 \in D(A^n)$, then the function $v(t)$ is n times continuously differentiable. Since $D(A^n)$ is dense in H, for the proof of the theorem it suffices to establish estimate (9). To prove the estimate $\|v(t)\|_{C^\alpha} \le M_\alpha \|v_0\|_H$ one must use the spectral decomposition

$$A = \int_\delta^{+\infty} \lambda \, dE_\lambda, \qquad \delta > 0$$

and the representation, following from it, for the function $\|v(t) - v(\tau)\|_H^2$

$$\|v(t) - v(\tau)\|_H^2 = \int_\delta^{+\infty} [e^{-t\lambda} - e^{-\tau\lambda}]^2 \lambda^{-2\alpha} \, d(E_\lambda v_0, v_0)_H.$$

Estimation of the integrand leads to the required inequality. Consider the integral

$$I = \int_0^T \int_0^T \|v(t) - v(\tau)\|_H^2 |t - \tau|^{-1-2\beta} \, dt \, d\tau =$$

$$= \int_\delta^{+\infty} \left[\int_0^T \int_0^T |e^{-t\lambda} - e^{-\tau\lambda}|^2 \lambda^{-2\alpha} |t - \tau|^{-1-2\beta} \, dt \, d\tau \right] d(E_\lambda v_0, v_0).$$

Further,

$$I = \int_\delta^{+\infty} \left[\int_0^{T\lambda} \int_0^{T\lambda} |e^{-s} - e^{-z}|^2 |s - z|^{-1-2\beta} \, ds \, dz \right] d(E_\lambda v_0, v_0) \le M_\alpha^2 \|v_0\|_H^2,$$

since the inner integrals are bounded uniformly in λ.

Corollary 1. *The operators AQ and $\dfrac{d}{dt}Q$ act boundedly from*

$$W_2^{1/2} \cap C^0 \quad \text{to} \quad W_2^{1/2}.$$

Theorem 5. *The operator $A^\beta Q$ acts boundedly from W_2^α to $C^{1/2+\alpha-\beta}$ for $0 \le \alpha < 1/2$ and $0 \le \beta \le 1/2 + \alpha$. The operator $A^\beta Q$ acts boundedly from $W_2^{1/2} \cap C^0$ to $C^{1-\beta}$ for $0 \le \beta \le 1$.*

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Proof. If $\alpha < \frac{1}{2}$ and $\beta < \frac{1}{2} + \alpha$, then the result follows from the embedding of the space W_2^α into the space $L_{2/(1-2\alpha)}$ and from the estimate

$$\|A^\alpha[\exp\{-(t+\Delta t)A\} - \exp\{-tA\}]\|_{H\to H} \le C(\alpha,\beta)\Delta t^\beta t^{-\alpha-\beta}e^{-\delta t}$$
$$(0 < t < t+\Delta t, \quad 0 \le \beta \le 1, \quad \alpha > -\beta), \tag{10}$$

which follows from the integral representation standing under the sign of the $\|\cdot\|_{H\to H}$ operator norm.

If, however, $\beta = \frac{1}{2} + \alpha$ and $\alpha < \frac{1}{2}$, then to study the operator $A^\beta Q$ we apply the methods of Sec. 1. Let v_0 be an arbitrary element of H. Put

$$\varphi(t) = (A^\beta Q f(t), v_0)_H = \int_0^t (A^\beta \exp\{-(t-s)A\} f(s), v_0)_H \, ds$$
$$= \int_0^t (f(s), A\exp\{-(t-s)A\}A^{\beta-1}v_0)_H \, ds.$$

Here the self-adjointness of the operator A has been used. Further,

$$\varphi(t) = \int_0^t \left(f(s), \frac{d}{ds}\left[\exp\{-(t-s)A\}A^{\beta-1}v_0\right]\right)_H ds = \int_0^t (f(s), \psi'(s))_H \, ds.$$

Let $\psi_1(s) = \psi(s) - \psi_2(s)$, where $\psi_2(s) = (s/t)[\psi(t) - \psi(0)]$. Then $\psi_1(s)$ is already periodic on $[0,t]$. Let also $f(s)$ be periodic on $[0,t]$. Then for the function

$$\varphi_1(t) = \int_0^t (f(s), \psi_1'(s))_H \, ds$$

the estimate $|\varphi_1(t)| \le \|f\|_{W_2^\alpha}\|\psi_1\|_{W_2^{1-\alpha}}$ is valid at the point t. Here the spaces W_2 are considered on the segment $[0,t]$. Further, $\|\psi\|_{W_2^{1-\alpha}} \le C_1\|v_0\|_H$. Here Theorem 4 is applied. With the aid of (10) we obtain $\|\psi_2\|_{W_2^{1-\alpha}} \le C_2\|v_0\|_H$. Therefore $\|\psi_1\|_{W_2^{1-\alpha}} \le C_3\|v_0\|_H$. Hence it follows that $|\varphi_1(t)| \le C_4\|f\|_{W_2^\alpha}\|v_0\|_H$. By virtue of (10),

$$|\varphi(t) - \varphi_1(t)| \le C_5 t^{-1/2-\alpha}\int_0^t \|f(s)\|_H \, ds \le C_6\|f\|_{L_{2/(1-2\alpha)}} \le C_7(\alpha)\|f\|_{W_2^\alpha}.$$

In the last inequality the embedding of W_2^α in $L_{2/(1-2\alpha)}$ has been used. Thus, finally, we obtain that $|\varphi(t)| \le M(\alpha)\|f\|_{W_2^\alpha}\|v_0\|_H$. By means of passage to the

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limit we free ourselves (since $\alpha < \frac{1}{2}$) from the assumption of periodicity of $f(s)$ on $[0, t]$.

Let now $\alpha = \frac{1}{2}$ and $\beta = 1$. Then the estimate of the function $AQf(t)$ is established as above in the case $\beta = \frac{1}{2} + \alpha$ with $\alpha < \frac{1}{2}$, but first one has to pass to a periodic function. From the estimate of the function $AQf(t)$ follows the estimate of the function $\frac{d}{dt}Qf(t)$, since the function $Qf(t)$ satisfies equation (3). Hence, from the multiplicative inequality for Hölder norms, the second assertion of Theorem 5 follows.

Corollary 2. The operators AQ and $\frac{d}{dt}Q$ act boundedly in the space $W_2^{1/2} \cap C^0$.

Theorem 6. In order that problem (1) have such a solution $v(t)$ that the functions $v'(t)$ and $Av(t)$ belong to the space $W_2^{1/2} \cap C^0$, it is necessary and sufficient that v_0 belong to $D(A)$ and $f(t)$ belong to $W_2^{1/2} \cap C^0$.

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Note: Figure translations are in progress. See original paper for figures.

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