V. G. Romanov

1968

SovietRxiv

Abstract

Full Text

V. G. Romanov

ON ONE FORMULATION OF AN INVERSE PROBLEM FOR A GENERALIZED WAVE EQUATION

(Presented by Academician A. N. Tikhonov on 23 XI 1967)

1.

Consider the equation for the function $u(M,t)$

$$n^2(M)\partial^2 u/\partial t^2 = \Delta u + a(M)u + f(M,t) \tag{1}$$

under the initial conditions

$$u(M,0) = \frac{\partial}{\partial t}u(M,0) = 0. \tag{2}$$

Here $M(x,y,z)$ is a point of three-dimensional space; Δ is the Laplace operator with respect to the variables x, y, z; $n(M)$ and $f(M,t)$ are given functions. Below we give a possible formulation of the problem of reconstructing the coefficient $a(M)$ from known functionals of the solution of equation (1) under conditions (2). Some results for this problem in the case $n(M) \equiv 1$ are contained in papers ([1]–[3]).

In what follows we shall assume that the function $f(M,t)$ has the form

$$f(M,t) = 4\pi\delta(t)\delta(M - M_0), \tag{3}$$

where $M_0(x_0, y_0, z_0)$ is some fixed point, and $\delta(t)$ is the Dirac delta function. In accordance with this, we denote the solution of the Cauchy problem posed for equation (1) by $u(M, M_0, t)$. We shall study some features of the connection between the functions $a(M)$ and $u(M, M_0, t)$.

Let the function $n(M)$ be twice continuously differentiable and such that to each pair of points $M(x,y,z)$, $P(\xi,\eta,\zeta)$ there corresponds one and only one geodesic $\Gamma(P,M)$ in the metric defined by the formula

$$ds = n(x,y,z)\sqrt{dx^2 + dy^2 + dz^2}, \tag{4}$$

and let, furthermore, $\tau(P,M)$ be the fundamental function of the central field of characteristics with center at the point M. The physical meaning of $\tau(P,M)$ is the time in which a disturbance produced at the point M reaches the point P. Some additional conditions on the function $n(M)$ will be imposed below. We shall regard the coefficient $a(M)$ as a continuous function of the point M.

Using S. L. Sobolev' s formula, we find that the solution of equation (1) under conditions (2) satisfies in this case the equation

$$u(M,M_0,t)=\frac{1}{4\pi}\iiint_{\tau(P,M)\le t} f(P,t-\tau(P,M))\sigma(P,M)\,dv_P+$$

$$+\frac{1}{4\pi}\iiint_{\tau(P,M)\le t}[\Delta_P\sigma(P,M)+a(P)\sigma(P,M)]\,u(P,M_0,t-\tau(P,M))\,dv_P, \quad (5)$$

in which the subscript P at the Laplace operator and the volume element dv means that, when they are evaluated, the variable point is the point $P(\xi,\eta,\zeta)$, while $\sigma(P,M)$ is a function determined by the function $n(M)$, having a singularity at the point M and satisfying some additional conditions (see ([4])). Since the function $f(M,t)$ is given by formula (3), the domain of integration in formula (5) degenerates into the domain bounded by the qua-

by the ellipsoid

$$S_{M,M_0,t}\{\tau(P,M_0)+\tau(P,M)=t\}$$

with foci at the points M_0, M. Indeed, at time t at the point M, the function $u(M,M_0,t)$ is affected only by those points P for which the sum of the travel times of the disturbances from the point M_0 to the point P and from it to the point M does not exceed t. Using formulas (3) and (5), for the function

$$v(M,M_0,t)=u(M,M_0,t)-\sigma(M,M_0)\delta(t-\tau(M,M_0)) \quad (6)$$

we obtain the equation

$$v(M,M_0,t)=\frac{1}{4\pi}\iiint\limits_{D_{M,M_0,t}}[\Delta_P\sigma(P,M)+a(P)\sigma(P,M)]\times$$

$$\times[\sigma(P,M_0)\delta(t-\tau(P,M_0)-\tau(P,M))+v(P,M_0,t-\tau(P,M))]\,dv_P, \quad (7)$$

where $D_{M,M_0,t}$ is the domain bounded by the quasiellipsoid $S_{M,M_0,t}$.

Consider, along with the quasiellipsoid $S_{M,M_0,t}$, the family of confocal quasiellipsoids $S_{M,M_0,T}$ $(T \geq \tau(M, M_0))$. We shall assume that the function $n(M)$ is such that, for a fixed point M, the point P can be uniquely determined, at least for t close to $\tau(M, M_0)$, by the curvilinear coordinates T, τ, φ, whose meaning is as follows. The coordinate T determines the quasiellipsoid on which the point P lies, $\tau = \tau(P, M_0)$, and φ is the angle of the spherical coordinate system characterizing the direction of the tangent at the point M_0 to the geodesic $\Gamma(P, M_0)$, if the polar axis of the system is aligned at the point M_0 with the direction of the tangent to the geodesic $\Gamma(M, M_0)$. In this case formula (7) can be written in the form

$$v(M, M_0, t) = \frac{1}{4\pi} \iint\limits_{S_{M,M_0,t}} [\Delta_P \sigma(P, M) + a(P)\sigma(P, M)]\sigma(P, M_0)\times$$

$$\times \left|\frac{\partial(\xi, \eta, \zeta)}{\partial(t, \tau, \varphi)}\right| d\tau\, d\varphi + \frac{1}{4\pi} \int\limits_{\tau(M,M_0)}^{t} dT \iint\limits_{S_{M,M_0,T}} [\Delta_P \sigma(P, M)+$$

$$+a(P)\sigma(P, M)] \left|\frac{\partial(\xi, \eta, \zeta)}{\partial(T, \tau, \varphi)}\right| v(P, M_0, t - \tau(P, M))\, d\tau\, d\varphi, \tag{8}$$

where $\partial(\xi, \eta, \zeta)/\partial(T, \tau, \varphi)$ is the Jacobian of the transformation from the Cartesian coordinates of the point P to the curvilinear ones.

Using the properties of the function $\sigma(P, M)$ and the assumptions made concerning the function $n(M)$, one can prove that the functions

$$R(P, M, M_0, t) = \tau(M, M_0)\sigma(P, M_0)\sigma(P, M) \left|\partial(\xi, \eta, \zeta)/\partial(t, \tau, \varphi)\right|,$$

$$Q(P, M, M_0, t) = \tau(M, M_0)\sigma(P, M_0)\Delta_P \sigma(P, M) \left|\partial(\xi, \eta, \zeta)/\partial(t, \tau, \varphi)\right|$$

are continuous in the aggregate of their arguments. Letting in equation (8) the argument t tend to $\tau(M, M_0)$ and taking into account that the integral over the volume bounded by the quasiellipsoid then tends to zero, we find

$$v(M, M_0, \tau(M, M_0)) = \frac{1}{2\tau(M, M_0)} \int\limits_{\Gamma(M,M_0)} [a(P)R(P, M, M_0, \tau(M, M_0))+$$

$$+Q(P, M, M_0, \tau(M, M_0))]\, d\tau, \tag{9}$$

where $d\tau = n(P)\,ds$, and ds is the element of length of the arc of the geodesic $\Gamma(M, M_0)$.

Up to now we have considered equation (1) in unbounded space. We now consider the equation in a domain bounded by a closed surface S, at each point of which there exists a tangent plane, and impose on it, in addition to conditions (2), also the boundary condition

$$\partial u/\partial n|_S = 0, \tag{10}$$

where $\mathbf{n}$ is the outward normal to S. Let M_0 be a point on the surface S and

$$f(M, t) = 2\pi\delta(t)\delta(M - M_0). \tag{11}$$

Then for the function $v(M, M_0, t)$, defined by formula (6), equality (9) also holds.

From formula (9) there follows the following possible formulation of the inverse problem: the value $v(M, M_0, \tau(M, M_0))$ is known as a function of a pair of points M, M_0 on the surface S; it is required to find the function $a(M)$.

Since the functions R and Q are known, the problem of reconstructing the function $a(M)$ in the indicated formulation is a problem of integral geometry.

In the case $n(M) = \text{const}$, all restrictions imposed on the function $n(M)$ are satisfied. The geodesic lines $\Gamma(M, M_0)$ in this case are straight lines joining the points M, M_0, and we have a well-studied ([5]) problem of integral geometry. From the results contained in the book ([5]) there follows the theorem:

Theorem 1. *For $n(M) = \text{const}$, the coefficient $a(M)$ of equation (1) in the class of continuous functions is uniquely reconstructed from the function $v(M, M_0, \tau(M, M_0))$.*

In the case of an arbitrary function $n(M)$, uniqueness of reconstruction of the function $a(M)$, of course, will not hold. However, in a number of cases uniqueness of reconstruction also holds for a variable function $n(M)$. On the basis of the results of article ([6]), the following theorem can be proved:

Theorem 2. *Let the function $n(M)$ satisfy all the conditions indicated in the text and, in addition, the following conditions:*

1) there exists such a finite point M^* that $n(M)$ is a function only of the distance to the point M^* (or $n(M)$ depends only on one coordinate, which corresponds to an infinitely remote point M^*);

2) the vertices of the curves $\Gamma(M, M_0)$ (i.e., the points on the geodesics $\Gamma(M, M_0)$ least distant from the point M^*) densely (in the sense of ([6])) fill the whole domain bounded by the surface S.

Then the reconstruction of the continuous function $a(M)$ from the function $v(M, M_0, \tau(M, M_0))$ is unique.

We note that, since the curves $\Gamma(M, M_0)$ corresponding to Theorems 1, 2 are plane curves, to reconstruct the function $a(M)$ it is in fact necessary to know the function $v(M, M_0, \tau(M, M_0))$ as a function of 3 variables, and not 4, as might seem. Indeed, it is enough to place the points M, M_0 on the contours of sections of the surface S by a one-parameter family of planes.

II. Analogously to the preceding discussion, one can pose the problem of reconstructing the function $a(M)$ for the equation

$$\partial^2 u/\partial t^2 = Lu + a(M)u + f(M,t), \tag{12}$$

where $M(x_1, x_2, x_3)$ is a point of a three-dimensional domain bounded by a smooth contour S; L is a given elliptic operator of the form

$$Lu \equiv \sum_{i=1}^{3} \left(a_i(M) \frac{\partial^2 u}{\partial x_i^2} + b_i(M) \frac{\partial u}{\partial x_i} \right) + h(M)u, \tag{13}$$

and $f(M,t)$ is defined by formula (11). Then, under conditions (2), (10), for the function $v(M, M_0, t)$, defined by formula (6), formula (9) can be obtained. The function $\sigma(P, M)$ here, of course, has a different meaning: it is expressed in terms of the coefficients $a_i(M)$, $b_i(M)$ $(i = 1, 2, 3)$ of the operator L.

For $a_i(M) = c_i$ $(i = 1, 2, 3)$, where c_i are constants, and under some smoothness conditions on the coefficients $b_i(M)$ $(i = 1, 2, 3)$, a theorem analogous to Theorem 1 holds.

In the case where $a_i(M), b_i(M)$ $(i = 1, 2, 3)$ depend only on the distance to some point M^* and the second of the conditions of Theorem 2 is fulfilled, so-

there is also uniqueness of the reconstruction of the coefficient $a(M)$ in the class of continuous functions.

Computing Center
of the Siberian Branch of the Academy of Sciences of the USSR

Received
21 VI 1967

REFERENCES

[1] Yu. M. Berezanskii, Tr. Mosk. matem. obshch., **7** (1958).
[2] M. M. Lavrent' ev, V. G. Romanov, DAN, **171**, No. 6, 1279 (1966).
[3] V. G. Romanov, DAN, **173**, No. 4 (1967).
[4] S. L. Sobolev, Tr. Seismological Institute, No. 6, 1 (1930).
[5] I. M. Gel' fand, M. I. Graev, N. Ya. Vilenkin, *Integral Geometry and Related*

Questions of Representation Theory, Moscow, 1962.
[6] V. G. Romanov, Sibirsk. matem. zhurn., **8**, No. 5, 1206 (1967).

Note: Figure translations are in progress. See original paper for figures.