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1968

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Abstract

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PARTITIONS OF LEBESGUE SPACE INTO TRAJECTORIES DETERMINED BY ER- GODIC AUTOMORPHISMS

R. M. Belinskaya

§ 1. Introduction

Let M be a Lebesgue space with continuous measure μ , and let T be its automorphism. Denote by τ_T the partition of the space M into the trajectories of the automorphism T . It is natural to call a property of the automorphism T *geometric* if every automorphism S possesses it for which the partition τ_S is isomorphic (mod 0) to τ_T . For example, periodicity, aperiodicity, and ergodicity are geometric properties. The main result of the present work (Theorem 2.7) is that, whatever the ergodic automorphisms S and T may be, the partitions τ_S and τ_T are isomorphic (mod 0). In particular, neither mixing, nor spectrum, nor entropy is a geometric characteristic of an automorphism. All of them depend essentially on the character of the *motion*, i.e., on the order in which the points of the trajectories are traversed.

From the point of view of measure theory, the theorem formulated above represents a certain advance in the problem of the metric classification of partitions of a Lebesgue space. The peculiarity of this advance is that it concerns nonmeasurable partitions (in the ergodic case the partition τ_T is always nonmeasurable). While for measurable partitions there has long been a complete metric classification (see [2]), tools that make it possible, at least in the simplest cases, to establish the isomorphism of nonmeasurable partitions were found only quite recently: they were indicated by A. M. Vershik [6], on whose results the present work essentially relies. One may hope that the same methods will make it possible to classify other nonmeasurable partitions encountered in ergodic theory, and it is not excluded that it will be possible to construct a meaningful general theory of sufficiently regular nonmeasurable partitions, independent of ergodic theory.

The results described above are collected in § 2. In §§ 3 and 4 the set \mathfrak{A}_τ of automorphisms T for which $\tau_T = \tau$ is studied. In § 3 sufficient conditions are established for the isomorphism of automorphisms with generating partitions into trajectories. In § 4 the uniform topology induced in \mathfrak{A}_τ from the space \mathfrak{A} of all automorphisms of the space M is studied. In particular, the separability of the space \mathfrak{A}_τ is established (recall that the space \mathfrak{A} itself is nonseparable). In the same section it is proved that if T is an ergodic automorphism, then the set of automorphisms conjugate to the given ergodic one is dense in the space \mathfrak{A}_{τ_T} ,

and on the basis of this theorem the location in \mathfrak{A}_τ of certain classes of automorphisms is studied: automorphisms with purely continuous spectrum, mixing automorphisms, and automorphisms with zero entropy. The results obtained are analogous to known theorems on the space \mathfrak{A}_a of aperiodic automorphisms (see [1]). The weak topology in \mathfrak{A}_τ

does not coincide with the uniform one. In the weak topology the set \mathfrak{A}_τ is everywhere dense in \mathfrak{A} .

Definitions of the basic concepts used in the paper may be found in [4] or in [1].

Taking this opportunity, the author warmly thanks V. A. Rokhlin for proposing the problems and for his constant attention to this work.

§ 2. Isomorphism of partitions on trajectories of ergodic automorphisms

2.1. Let $D \subset M$, $\mu D > 0$. The automorphism T'_D of the subspace D of the space M , defined by the equality

$$T'_D x = T^{p(x)} x, \quad x \in D,$$

where $p(x)$ is the least of the natural numbers l such that $T^l x \in D$, is called the *derived* automorphism of T on the set D . By $\varphi(T; D; \cdot)$ we denote the function defined by the equality $\varphi(T; D; x) = p(x)$, $x \in D$. If the set D is fixed, then we shall write $\varphi_T(x)$ instead of $\varphi(T; D; x)$.

2.2. Lemma. *Let S and T be automorphisms of the space M , with $\tau_S \geq \tau_T$. If there exists a set $D \subset M$, $\mu D > 0$, such that $S'_D = T'_D$ and*

$$\bigcup_{k=0}^{\infty} S^k D = M,$$

then $\tau_S = \tau_T$.

Proof. If one assumes that $\tau_S \neq \tau_T$, then, since $S'_D = T'_D$, it is necessary to suppose that there is an element of the partition τ_S lying entirely in $M \setminus D$, which contradicts the equality

$$\bigcup_{k=0}^{\infty} S^k D = M.$$

2.3. Lemma. *If T is an ergodic automorphism, then for any sets A and B of equal positive measure there exists a mapping I of the set A onto the set B such that $Ix = T^{\psi(x)} x$ ($x \in A$), where $\psi(\cdot)$ is a positive integer-valued measurable function defined on the set A .*

Proof. Put $B_1 = TA \cap B$, $A_1 = T^{-1}B_1$, and

$$B_n = T^n \left(A \setminus \bigcup_{i=1}^{n-1} A_i \right) \cap \left(B \setminus \bigcup_{i=1}^{n-1} B_i \right), \quad A_n = T^{-n}B_n \quad \text{for } n = 2, 3, \dots$$

Since T is ergodic, at least one of the sets B_n has positive measure. We shall show that

$$\mu \left(B \setminus \bigcup_{n=1}^{\infty} B_n \right) = 0.$$

Indeed, if

$$\mu \left(B \setminus \bigcup_{n=1}^{\infty} B_n \right) > 0,$$

then also

$$\mu \left(A \setminus \bigcup_{n=1}^{\infty} A_n \right) > 0,$$

and, by ergodicity of T , there is a number l such that

$$\mu \left[T^l \left(A \setminus \bigcup_{n=1}^{\infty} A_n \right) \cap \left(B \setminus \bigcup_{n=1}^{\infty} B_n \right) \right] > 0,$$

but then it would have to be that

$$T^l \left(A \setminus \bigcup_{n=1}^{\infty} A_n \right) \cap \left(B \setminus \bigcup_{n=1}^{\infty} B_n \right) \subset B_l,$$

which is impossible. Thus,

$$B = \bigcup_{n=1}^{\infty} B_n \quad (\text{mod } 0)$$

and, evidently,

$$A = \bigcup_{n=1}^{\infty} A_n \quad (\text{mod } 0).$$

Consequently, one may put $\psi(x) = n$ for $x \in A_n$, and $Ix = T^{\psi(x)}x$ for $x \in A$.

A mapping I with the indicated properties is not unique. Any such mapping will be called *compatible* with T .

2.4. Lemma. Let T be an aperiodic automorphism. Then there exists a set B of measure $1/2$ such that the automorphism T'_B has among its eigenvalues all roots of unity of degree 2^i , $i = 1, 2, \dots$

Proof. From the aperiodicity of T it follows that there exists a set A_1 such that $\mu A_1 = 3/8$ and $A_1 \cap T A_1 = \Lambda$ (see [1]). Put $A_1^1 = A_1$, $A_1^2 = T A_1$, $B_1 = A_1^1 \cup A_1^2$; then $\mu B_1 = 3/4$. The automorphism T'_{B_1} has -1 as an eigenvalue. Suppose sets B_2, \dots, B_{n-1} have been found such that for $i = 1, 2, \dots, n-1$

$$1) B_i \supset B_{i+1}, \quad 2) \mu B_i = (2^i + 1)/2^{i+1}, \quad 3) B_i = \bigcup_{k=1}^{2^i} A_i^k,$$

$$4) T'_{B_i} A_i^k = A_i^{k+1}, \quad k = 1, 2, \dots, 2^i - 1; \quad T'_{B_i} A_i^{2^i} = A_i, \quad 5) A_i \subset A_{i-1}.$$

Using the aperiodicity of $T'_{A_{n-1}}$, find a set $A_n \subset A_{n-1}$ such that

$$\mu_{A_{n-1}} A_n = \frac{2^n + 1}{2^{n+1} + 4}$$

and $T'_{A_{n-1}} A_n \cap A_n = \Lambda$ (for any set $F \subset A_{n-1}$ we have $\mu_{A_{n-1}} F = \frac{\mu F}{\mu A_{n-1}}$). Put $A_n^1 = A_n$, $A_n^k = T'_{B_{n-1}} A_n^{k-1}$, $k = 2, 3, \dots, 2^n$;

$$B_n = \bigcup_{k=1}^{2^n} A_n^k.$$

Then

$$\begin{aligned} \mu B_n &= 2^n \mu A_n = 2^n \mu_{A_{n-1}} \mu_{A_{n-1}} A_n = 2 \cdot 2^{n-1} \mu_{A_{n-1}} \mu_{A_{n-1}} A_n = \\ &= 2 \mu_{B_{n-1}} \mu_{A_{n-1}} A_n = 2 \cdot \frac{2^{n-1} + 1}{2^n} \cdot \frac{2^n + 1}{2^{n+1} + 4} = \frac{2^{n-1} + 1}{2^n} \cdot \frac{2^n + 1}{2^n + 2} = \frac{2^n + 1}{2^{n+1}}. \end{aligned}$$

Continuing this process, put

$$B = \bigcap_{n=1}^{\infty} B_n, \quad \mu B = 1/2.$$

It is clear from the construction that for any $i > 0$ there is a sequence $\{\tilde{A}_i^k\}_{k=1}^{2^i}$ ($\tilde{A}_i^k = A_i^k \cap B$) such that

$$B = \bigcup_{k=1}^{2^i} \tilde{A}_i^k, \quad T'_B \tilde{A}_i^k = \tilde{A}_i^{k+1}, \quad k = 1, \dots, 2^i - 1;$$

$$T'_B \tilde{A}_i^{2^i} = \tilde{A}_i^1,$$

and moreover $\tilde{A}_i^1 \subset \tilde{A}_{i-1}^1$. Hence it follows that T'_B has as an eigenvalue a root of unity of degree 2^i .

2.5. Lemma. Let τ be a partition such that the automorphisms of the set \mathfrak{A}_τ are ergodic. Then in \mathfrak{A}_τ there exists an automorphism R among whose eigenvalues there are all roots of unity of degree 2^i , $i = 1, 2, \dots$

Proof. Let $T \in \mathfrak{A}_\tau$, and let B be a set of measure $1/2$ such that the automorphism T'_B has among its eigenvalues all roots of unity of degree 2^i , $i = 1, 2, \dots$, (Lemma 2.4), and let I be a mapping of the set B onto the set $M \setminus B$, compatible with T (see § 2.3). Define the automorphism R by setting

$$Rx = \begin{cases} Ix, & \text{if } x \in B, \\ T'_B I^{-1}x, & \text{if } x \in M \setminus B. \end{cases}$$

Obviously, $\tau_R \geq \tau_T$. Since $R'_B = T'_B$, by Lemma 2.2, $\tau_R = \tau_T$.

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The automorphism R has as its eigenvalue every 2^i -th root of unity, $i = 1, 2, \dots$. Indeed, put $\tilde{A}_0^1 = B$, $D_i^{2^k-1} = \tilde{A}_{i-1}^k$, $D_i^{2^k} = R\tilde{A}_{i-1}^k$, $k = 1, \dots, 2^{i-1}$. It is clear that

$$M = \bigcup_{k=1}^{2^i} D_i^k, \quad RD_i^k = D_i^{k+1},$$

$k = 1, \dots, 2^k - 1$; $RD_i^{2^i} = D_i^1$, as required.

2.6. This subsection contains the information needed for the formulation and proof of Theorem 2.7. We shall consider arbitrary (i.e., generally speaking, nonmeasurable) partitions of the space M .

Two partitions ξ and η are called isomorphic mod 0 if there exist sets A and B of measure 0 and an isomorphism U of the spaces $M \setminus A$ and $M \setminus B$ such that the partitions ξ' and η' , induced by the partitions ξ and η in the spaces $M \setminus A$ and $M \setminus B$, respectively, are related by the equality $U\xi' = \eta'$.

Let ξ_1, ξ_2, \dots be a finite or infinite sequence of partitions of the space M . By $\bigcap_i \xi_i$ we denote the following partition: points x and y belong to the same element of the partition if and only if there exists a finite sequence of points x, x_1, \dots, x_n, y such that each two neighboring points belong to one and the same element of one of the partitions ξ_i . In the case of a finite number of partitions we shall also write $\xi_1 \cap \xi_2 \cap \dots \cap \xi_n$. This partition, generally speaking, is nonmeasurable even if all the partitions ξ_i are measurable.

A measurable partition ξ , each element of which consists of k points with conditional measures $1/k$, will be called a k -partition of the space M . From the classification of measurable partitions (see [2]) it follows that all k -partitions with fixed k are isomorphic to one another.

The following theorem belongs to A. M. Vershik (see [6]):

Suppose there are two sequences of partitions:

$$\xi_1 > \xi_2 > \dots, \quad \xi_i \searrow \nu,$$

$$\eta_1 > \eta_2 > \dots, \quad \eta_i \searrow \nu,$$

where ξ_i and η_i are 2^i -partitions. Then the partitions $\tau_1 = \bigcap_{i=1}^{\infty} \xi_i$ and $\tau_2 = \bigcap_{i=1}^{\infty} \eta_i$ are isomorphic.

2.7. Theorem. *If T and S are ergodic automorphisms, then τ_T and τ_S are isomorphic.*

Proof. Let $R \in \mathfrak{A}_{\tau_T}$, and let R have among its eigenvalues all 2^i -th roots of unity, $i = 1, 2, \dots$ (see Lemma 2.5), and let $\{D_i^k\}_{k=1}^{2^i}$ be a system of sets such that

$$M = \bigcup_{k=1}^{2^i} D_i^k, \quad RD_i^k = D_i^{k+1}$$

for $k = 1, \dots, 2^i - 1$ and $RD_i^{2^i} = D_i^1$. Construct a sequence of 2^i -partitions $\{\xi_i\}_{i=1}^{\infty}$, assigning to one element of the partition ξ_i , together with each point $x \in D_i^1$, the points $Rx, R^2x, \dots, R^{2^i-1}x$, and only them. We shall show that

$$\bigcap_{i=1}^{\infty} \xi_i = \tau_R \pmod{0}.$$

Let $R^n x = y$ for $n > 0$ ($x, y \in M$) and $i > \lg n$. If the points x and y do not lie in one element of the partition ξ_i , then there is a unique number l such that $0 < l < n$ and $R^l x \in D_i^1$. Choose a number j such that $R^l x \in D_j^1$ (since

$\lim_{i \rightarrow \infty} \mu D_i^1 = 0$, for almost all $x \in M$ such a number j will be found). Then the points x and y

will fall into one element of the partition ξ_j . Thus, $\tau_T = \bigcap_{i=1}^{\infty} \xi_i \pmod{0}$. Similarly, we find a sequence of 2^i -partitions $\{\eta_i\}_{i=1}^{\infty}$ such that $\tau_S = \bigcap_{i=1}^{\infty} \eta_i \pmod{0}$. By Vershik's theorem it follows that the partitions τ_T and τ_S are isomorphic $\pmod{0}$.

2.8. Corollary. *Mixing, spectrum, entropy, and the K -property are not geometric characteristics of an automorphism.*

2.9. In addition to Theorem 2.7, below we shall give a characterization of the partitions determined by the trajectories of ergodic automorphisms. This and the following subsections contain preliminary information needed for the formulation and proof of Theorem 2.11.

If P is an automorphism of the space M having period 2, then, evidently, τ_P is a 2-partition. It is also clear that for every 2-partition ζ of the space M there exists a unique automorphism P of period 2 such that $\tau_P = \zeta$.

We shall call a partition τ *completely nonmeasurable* if every measurable set consisting of whole elements of the partition τ has either measure 0 or measure 1 (it is obvious that a completely nonmeasurable partition is nonmeasurable).

It is clear that an automorphism T is ergodic if and only if the partition τ_T is completely nonmeasurable.

2.10. Lemma. *For any two 2-partitions ξ and η there exists one and only one partition α possessing the following properties:*

- 1) $\alpha \cap \xi = \alpha \cap \eta = \xi \cap \eta$,
- 2) *if an element of the partition ξ (or η) and an element of the partition α are contained in one element of the partition $\xi \cap \eta$, then their intersection is nonempty and consists of one point.*

This partition coincides with the partition τ_{PQ} , where P and Q are the automorphisms of period 2 determined by the partitions ξ and η , respectively (see 2.9).

We shall denote the partition α with the indicated properties by $\xi \circ \eta$.

Proof. It is obvious that the partition τ_{PQ} has the required properties. Let α be an arbitrary partition possessing the listed properties. We shall prove that $\alpha = \tau_{PQ}$.

From the properties of the partition α it follows that each element of the partition $\xi \cap \eta$ consists of two elements of the partition α . Since, on the one hand, x and Qx , and, on the other hand, Qx and PQx , must lie in different elements of the partition α , but in one element of the partition $\xi \cap \eta$, the points x and PQx lie in one element of the partition α , whence $\alpha \leq \tau_{PQ}$. But since $\tau_{PQ} \geq \xi \cap \eta$

and each element of the partition $\xi \cap \eta$ contains two elements of the partition τ_{PQ} , it follows that $\alpha = \tau_{PQ}$.

2.11. Theorem. *The class of partitions τ_T , where T is an ergodic automorphism, coincides with the class of completely nonmeasurable partitions of the form $\xi \cap \eta$, where ξ and η are 2-partitions for which $\xi \circ \eta$ is not completely nonmeasurable.*

Proof. Let T be an ergodic automorphism and let $R \in \mathfrak{A}_{\tau_T}$ be the automorphism whose existence is asserted in Lemma 2.5, with $RD_1^1 = D_1^2$, $RD_1^2 = D_1^1$. Put

$$Px = \begin{cases} Rx, & \text{if } x \in D_1^1, \\ R^{-1}x, & \text{if } x \in D_1^2, \end{cases} \quad Qx = \begin{cases} R^{-1}x, & \text{if } x \in D_1^1, \\ Rx, & \text{if } x \in D_1^2. \end{cases}$$

It is obvious that the automorphisms P and Q have period 2 and $\tau_T = \tau_P \cap \tau_Q$. Since $PQD_1^1 = D_1^1$, the partition τ_{PQ} is not completely nonmeasurable.

Conversely, let $\tau = \xi \cap \eta$, and let $\xi \circ \eta$ be not completely nonmeasurable. We shall show that there exists an ergodic automorphism T for which $\tau_T = \tau$. Let P and Q be automorphisms of period 2 determined by the 2-partitions ξ and η , respectively (see 2.9). Then $\xi \circ \eta = \tau_{PQ}$ (Lemma 2.10). Since τ_{PQ} is not a completely nonmeasurable partition, the automorphism PQ is nonergodic. Let Γ be a set invariant with respect to the automorphism PQ . Every element of the partition τ having a nonempty intersection with Γ does not intersect the set $M \setminus (Q\Gamma \cup \Gamma)$. Since τ is completely nonmeasurable, the measure of the union of those invariant sets Γ for which $Q\Gamma \cup \Gamma \neq M \pmod{0}$ is 0. Hence it follows that if $\mu\Gamma > 0$, then $Q\Gamma \cup \Gamma = M \pmod{0}$ and $\mu\Gamma \geq 1/2$; and since the same is true for any set invariant with respect to the automorphism PQ , it follows that $\mu\Gamma = 1/2$ and $\mu(Q\Gamma \cap \Gamma) = 0$. Similarly one shows that $\mu(P\Gamma \cap \Gamma) = 0$. If T is defined by the equality

$$Tx = \begin{cases} Qx, & \text{if } x \in \Gamma, \\ Px, & \text{if } x \notin \Gamma, \end{cases}$$

then $\tau_T = \tau$.

§ 3. The function $\theta(T, S; \cdot)$ and its properties

3.1. In this section automorphisms are assumed to be aperiodic. Let T, S be automorphisms of the space M , and let $\tau_S \geq \tau_T$. Then there exists an integer-valued function $\theta(T, S; \cdot)$ on M such that $Sx = T^{\theta(T, S; x)}x$. We denote the function $\theta(T, S^k; \cdot)$ by $\theta_k(T, S; \cdot)$. It is clear that

$$\theta_k(T, S; x) = \sum_{i=0}^{k-1} \theta(T, S; S^i x) \quad \text{for } k > 0,$$

$$\theta_k(T, S; x) = - \sum_{i=1}^k \theta(T, S; S^{-i} x) \quad \text{for } k < 0,$$

and

$$\theta_k(T, S^{-1}; x) = \theta_{-k}(T, S; x) \quad (k = \pm 1, \pm 2, \dots).$$

In order that $\tau_S = \tau_T$, it is necessary and sufficient that the sequence

$$\{\theta_k(T, S; x)\}_{k=-\infty}^{\infty}$$

run through all integers for every $x \in M$ (we shall assume that $\theta_0(T, S; x) = 0$).

If $\tau_S = \tau_T$, then the function $\theta(S, T; \cdot)$ is also defined on M . From the definition it follows that

$$\theta(T, S^{\theta_k(S, T; x)}, x) = k.$$

Let T be an automorphism, and let $f(\cdot)$ be an arbitrary integer-valued function on M . In order that the mapping S of the space M onto itself, defined by the equality $Sx = T^{f(x)}x$, be an automorphism, it is necessary and sufficient that, for almost all $x \in M$, the equalities $f(x) - f(T^p x) = p$ and $p = 0$ be equivalent. Then $f(x) = \theta(T, S; x)$. Moreover, in order that $\tau_S = \tau_T$, it is necessary and sufficient that there exist on M a function $h(\cdot)$ such that $\theta(T, S^{h(x)}; x) = 1$ for almost all $x \in M$. Then $h(x) = \theta(S, T; x)$.

If the automorphisms T and S are fixed, then we shall write $\theta(x)$ instead of $\theta(T, S; x)$.

A numerical sequence all of whose terms, beginning with some point, are positive (negative) will be called *almost positive* (*almost negative*). Almost positive and almost negative sequences will also be called *almost sign-constant*.

3.2. Theorem. *If the function $\theta(T, S; \cdot)$ is integrable, then for almost every $x \in M$ the sequences $\{\theta_k(T, S; x)\}_{k=1}^{\infty}$ and $\{\theta_{-k}(T, S; x)\}_{k=1}^{\infty}$ are almost sign-constant. If the automorphism S is ergodic, then on the entire mod 0 space M the sign is one and the same.*

Proof. We adopt the following notation:

$$C_k = \{x \in M \mid \theta(x) = k\}, \quad k = 1, 2, \dots, \quad C^N = \bigcup_{k=N}^{\infty} \bigcup_{i=0}^{k-1} T^i C_k, \quad N = 1, 2, \dots,$$

$$F_{x_0} = \{x \in M \mid x = T^{-l} x_0, \quad l > 0; \theta(T, S; x) > l\}, \quad x_0 \in M.$$

If $x_0 \notin C^N$, then $0 < \theta(x) < N$ for $x \in F_{x_0}$, and the set F_{x_0} is finite. Then $\theta_k(x_0)$ preserves its sign for all $k > q$ and $k < q_1$, where

$$q = \max\{k > 0 \mid S^k x_0 \in F_{x_0}\},$$

$$q_1 = \min\{k < 0 \mid S^k x_0 \in F_{x_0}\}.$$

Since the function $\theta(\cdot)$ is integrable and

$$\mu C^N \leq \sum_{k=N}^{\infty} k \mu C_k,$$

we have $\lim \mu C^N = 0$. Consequently, the required property holds for almost every $x \in M$.

Now let the automorphism S be ergodic, and let A be the set of those $x \in M$ for which the sequence $\{\theta_k(x)\}_{k=1}^{\infty}$ is almost positive. Since $SA = A$, either $\mu A = 0$ or $\mu A = 1$.

Remark. Let

$$\theta^+(x) = \max\{\theta(T, S; x), 0\}, \quad \theta^-(x) = \max\{-\theta(T, S; x), 0\}.$$

The theorem is also true in the case where at least one of the functions $\theta^+(\cdot)$ or $\theta^-(\cdot)$ is integrable.

3.3. Lemma. *If, for almost all $x \in M$, the sequence*

$$\{\theta_k(T, S; x)\}_{k=1}^{\infty}$$

is almost positive, then there exists a set D such that: a) $\mu D > 0$; b) $\theta_k(T, S; x) > 0$, $k = 1, 2, \dots$, for all $x \in D$; c) if $\tau_T = \tau_S$, then $S'_D = T'_D$.

Proof. Let $\theta_k(T, S; x) > 0$ for $k > q$ (q depends on x). Put

$$r(x) = \max\{k > q \mid \theta_k(x) < \theta_q(x)\}$$

and

$$D = \{x \in M \mid \theta_k(x) > 0, \quad k = 1, 2, \dots\}.$$

Since $S^{r(x)} x \in D$ ($x \in M$), we have

$$\bigcup_{r=0}^{\infty} S^{-r} D = M$$

and $\mu D > 0$. If $\tau_S = \tau_T$, then, evidently, $\tau_{S'_D} = \tau_{T'_D}$. Since $\theta(T'_D, S'_D; x) > 0$ for all $x \in D$, it follows that

$$\theta(T'_D, S'_D; x) = 1$$

for all $x \in D$ and $S'_D = T'_D$.

3.4. Theorem. *If, for almost all $x \in M$, the sequence*

$$\{\theta_k(T, S; x)\}_{k=1}^{\infty}$$

(or the sequence

$$\{\theta_{-k}(T, S; x)\}_{k=1}^{\infty}$$

) is almost sign-constant and $\tau_T = \tau_S$, then the automorphisms T and S have the same entropy.

Proof. As is known (see [5]), the entropy of an automorphism T is related to the entropy of its derivative by the equality

$$h(T) = h(T'_D)\mu D,$$

and the entropies of the automorphisms T and T^{-1} are equal (see [4]). Hence the assertion of the theorem follows from Lemma 3.3.

3.5. Lemma. *If the sequence*

$$\{\theta_k(T, S; x)\}_{k=1}^{\infty}$$

is almost positive, while the sequence

$$\{\theta_{-k}(T, S; x)\}_{k=1}^{\infty}$$

is almost negative for almost all $x \in M$, then there exists a set $F \subseteq M$ such that $\mu F > 0$,

$$\theta_k(T, S; x) > 0 \quad (k = 1, 2, \dots)$$

and

$$\theta_{\varphi(S, F; x)}(x) \geq \varphi(S, F; x)$$

for all $x \in F$; if $\tau_T = \tau_S$, then $S'_F = T'_F$ (i.e.

$$\theta_{\varphi(S, F; x)}(x) = \varphi(T, F; x)$$

and

$$\varphi(S, F; x) = \varphi(T, F; x)$$

for all $x \in F$ (for the definition of the function φ , see 2.1).

Proof. Let D be the set indicated in Lemma 3.3. Denote by F_p the set of points $x \in D$ for which the sequence

$$\{\theta_{-k}(x)\}_{k=1}^{\infty}$$

has p positive terms ($p = 0, 1, 2, \dots$). Since $\theta_{-k}(x) < 0$ for all sufficiently large k ($x \in D$), we have

$$D = \bigcup_{p=0}^{\infty} F_p$$

and $\mu F_p > 0$ for some p . Fix such a p . Since

$$\theta_k(T, S; x) > 0 \quad (k = 1, 2, \dots)$$

for all $x \in D$, the same is true for all $x \in F_p$. Let $\tau_S = \tau_T$. Then, since...

$S'_D = T'_D$, then $S'_{F_p} = T'_{F_p}$, i.e. $\theta_{\varphi_S(x)}(x) = \varphi_T(x)$ ($x \in F_p$). We shall prove that for every $x \in F_p$ the equality $\varphi_T(x) = \varphi_S(x)$ holds.

Let $\theta_{-k_i}(x) = l_i$, $k_i, l_i > 0$, $i = 1, \dots, p$. Of these, let $l_{i_1}, \dots, l_{i_j} < \theta_{\varphi_S(x)}(x)$, $l_{i_{j+1}}, \dots, l_{i_p} > \theta_{\varphi_S(x)}(x)$; since $\theta_k(S'_{F_p} x) > 0$, $k = 1, 2, \dots$, for any l , $0 < l < \theta_{\varphi_S(x)}(x)$ and $l \neq l_{i_q}$ ($q = 1, \dots, j$), there is a number k such that $0 < k < \varphi_S(x)$ and $\theta_k(x) = l$. Since $S'_{F_p} x \in F_p$, there must be exactly $p - (p - j)$ numbers k such that $\theta_k(x) > \theta_{\varphi_S(x)}(x)$, $0 < k < \varphi_S(x)$. Thus the sequence $\theta_1(x), \dots, \theta_{\varphi_S(x)}(x)$ must consist of $(\theta_{\varphi_S(x)}(x) - j) + j$ numbers, whence $\theta_{\varphi_S(x)}(x) = \varphi_S(x)$ and, consequently, $\varphi_T(x) = \varphi_S(x)$. If $\tau_S \geq \tau_T$, then an analogous argument shows that $\theta_{\varphi_S(x)}(x) \geq \varphi_S(x)$ for all $x \in F_p$. Taking $F = F_p$, we see that F is the required set.

3.6. Theorem. Let $\tau_T = \tau_S$ for the automorphisms T and S , and suppose that there exists a set F such that

$$\bigcup_{k=0}^{\infty} S^{kF} = M, \quad T'_F = S'_F$$

and $\varphi(T; F; x) = \varphi(S; F; x)$ for $x \in F$. Then T and S are isomorphic.

Proof. By hypothesis, if $x, S^{kx} \in F$, then $\theta_k(x) = k$. Define an integer-valued function $g(\cdot)$ on M so that the mapping U of the space M onto itself, defined by the equality $Ux = T^{g(x)}x$, is an automorphism and $US = TU$. Put $g(x) = 0$ if $x \in F$, and let $g(Sx) = g(x) - \theta(x) + 1$. Then $g(S^{kx}) = g(x) - \theta_k(x) + k$. If $S^{kx} \in F$, then $\theta_k(x) = k$ and $g(S^{kx}) = g(x) = 0$. Consequently, the definition of the function $g(\cdot)$ is correct. To verify that U is an automorphism, it suffices to show that the equalities $g(x) = g(S^{px})$, $\theta_p(x) = p$ are equivalent for almost all $x \in M$ (see 3.1). This follows from the fact that

$$g(x) - g(S^{px}) = \theta_p(x) - p.$$

Let us check that $US = TU$. On the left,

$$USx = T^{g(Sx)}T^{\theta(x)}(x) = T^{g(Sx)+\theta(x)}x.$$

On the right,

$$TUx = T^{g(x)+1}x.$$

But $g(Sx) + \theta(x) = g(x) + 1$ by the definition of $g(\cdot)$.

3.7. Corollary. If the automorphisms T and S are such that $\tau_T = \tau_S$, and the sequence $\{\theta_k(T, S; x)\}_{k=1}^\infty$ is almost positive, while the sequence $\{\theta_{-k}(T, S; x)\}_{k=1}^\infty$ is almost negative for almost all $x \in M$, then T and S are isomorphic.

If the automorphisms T and S are ergodic, $\tau_T = \tau_S$, and the function $\theta(T, S; x)$ is integrable, then either T and S , or T and S^{-1} , are isomorphic.

Remark. From the fact that only one of the sequences $\{\theta_k(T, S; x)\}_{k=1}^\infty$, $\{\theta_{-k}(T, S; x)\}_{k=1}^\infty$ is almost sign-constant, the isomorphism of the automorphisms T and S , or of T and S^{-1} , still does not follow. Indeed, let T be an ergodic automorphism having no -1 as an eigenvalue, let A be any set of measure $1/2$, and let I be a mapping of the set $A \cap T^{-1}A$ onto the set $M \setminus (A \cup TA)$, compatible with T (see 2.3). Define the automorphism S by the equality

$$Sx = \begin{cases} Tx, & \text{if } x \in A \setminus T^{-1}A, \\ Ix, & \text{if } x \in A \cap T^{-1}A, \\ T'_A T^{-1}x, & \text{if } x \in TA \setminus A, \\ T'_A I^{-1}x, & \text{if } x \in M \setminus (A \cup TA). \end{cases}$$

Since $S'_A = T'_A$, we have $\tau_T = \tau_S$ (Lemma 2.2). It is clear from the construction that, for every $x \in A$, the sequence $\{\theta_k(T, S; x)\}_{k=1}^\infty$ is positive; therefore, for

for any $x \in M$ the sequence $\{\theta_k(T, S; x)\}_{k=1}^\infty$ is almost positive. But since $S^2A = A$, the automorphism S has -1 as an eigenvalue and, consequently, is not isomorphic to the automorphism T (note that $h(T) = h(S)$).

3.8. Theorem. Let T, S be ergodic automorphisms and let the function $\theta(T, S; x)$ be integrable. Then the relations $\tau_T = \tau_S$ and $|\int \theta(T, S; x) d\mu| = 1$ are equivalent.

Proof. From the ergodicity of S it follows that, for almost all $x \in M$,

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \sum_{i=0}^{k-1} \theta(S^i x) = \int \theta(x) d\mu.$$

In other words,

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \theta_k(x) = \int \theta(x) d\mu = c, \quad x \in M. \quad (*)$$

Since the function $\theta(\cdot)$ is integrable and the automorphism S is ergodic, by Theorem 3.2 the sequences $\{\theta_k(x)\}_{k=1}^\infty$ and $\{\theta_{-k}(x)\}_{k=1}^\infty$ are almost sign-constant, and the sign is the same almost everywhere on the whole space M . Suppose that $\theta_k(x) > 0$, $\theta_{-k}(x) < 0$ for almost all $x \in M$ for sufficiently large k . Let $\tau_T = \tau_S$. We shall prove that $c = 1$. By Lemma 3.5 there exists a set F such that $\mu F > 0$, $T_F = S_F$, and if $x, S^k x \in F$, then $\theta_k(x) = k$. Consequently, for any $x \in F$ there exists a sequence $\{\theta_{k_i}(x)\}_{i=1}^\infty$ such that

$$\frac{\theta_{k_i}(x)}{k_i} = 1.$$

Since $\lim_{k \rightarrow +\infty} \frac{\theta_k(x)}{k}$ exists, it is equal to 1 for any $x \in F$ and, by virtue of (*), for any $x \in M$.

Now suppose that $\tau_S \geq \tau_T$, but $\tau_S \neq \tau_T$, and prove that then $c > 1$. By Lemma 3.5 there exists a set F such that $\mu F > 0$ and, for all $x \in F$, $\theta_k(T, S; x) > 0$, $k = 1, 2, \dots$, and $\theta_{\varphi(S; F; x)}(x) \geq \varphi(S; F; x)$. Hence it follows that $\theta(T'_F, S'_F; x) > 0$ for any $x \in F$. Since S is ergodic, $\tau_{S'_F} \geq \tau_{T'_F}$, but $\tau_{S'_F} \neq \tau_{T'_F}$. Therefore, if

$$E = \{x \in F \mid \theta(T'_F, S'_F; x) \geq 2\},$$

then $\mu E > 0$. For any $x \in E$, we have $\theta_{\varphi(S; E; x)}(x) - \varphi(S; E; x) \geq 1$. If $x, S^{k_i} x \in E$, $k_i > 0$, then

$$\theta_{k_i}(T, S; x) - k_i \geq i$$

(the number i indicates how many times, under the action of S , the point x was in E in k_i steps); hence

$$\frac{\theta_{k_i}(T, S; x)}{k_i} - 1 \geq \frac{i}{k_i}.$$

From the ergodicity of S it follows that

$$\lim_{i \rightarrow \infty} \frac{i}{k_i} = \mu E.$$

Then

$$\lim_{i \rightarrow \infty} \frac{\theta_{k_i}(T, S; x)}{k_i} \geq 1 + \mu E$$

(the existence of the limit follows from (*)). Consequently,

$$\lim_{k \rightarrow \infty} \frac{\theta_k(T, S; x)}{k} \geq 1 + \mu E$$

for any $x \in E$, and, by (*), this is true for any $x \in M$. Thus,

$$\int \theta(x) d\mu > 1.$$

§ 4. The Space \mathfrak{A}_τ

4.1. The set \mathfrak{A} of all automorphisms of the space M is a complete metric space with the metric d defined by the equality

$$d(S, T) = \mu\{x \in M \mid Sx \neq Tx\}, \quad S, T \in \mathfrak{A}$$

(see [1]). This space is nonseparable. The topology defined by the metric d is called the uniform topology.

Let τ be a partition such that the automorphisms of the set \mathfrak{A}_τ are aperiodic. The metric induced on \mathfrak{A}_τ by the metric d turns it into a metric-space. It is clear that if $T, S \in \mathfrak{A}_\tau$, then

$$d(S, T) = \mu\{x \in M \mid \theta(T, S; x) \neq 1\}.$$

Otherwise, if $T, S, R \in \mathfrak{A}_\tau$, then

$$d(S, T) = \mu\{x \in M \mid \theta(R, S; x) \neq \theta(R, T; x)\}.$$

For fixed R the metric determined by the last equality will be denoted by d_R .

4.2. Theorem. *The space \mathfrak{A}_τ is not complete in the metric d . The space \mathfrak{A}'_τ of all automorphisms J for which $\tau_J \geq \tau$ is complete in this metric.*

Proof. The following example shows that there exists a sequence of automorphisms from \mathfrak{A}_τ converging in \mathfrak{A} in the metric d , whose limiting automorphism does not belong to the space \mathfrak{A}_τ .

Let $T \in \mathfrak{A}_\tau$, and let $\{E_n\}_{n=1}^\infty$ be a sequence of sets such that

$$E_n \supset E_{n+1} \quad (n = 1, 2, \dots)$$

and

$$\mu \left(\bigcap_{n=1}^{\infty} E_n \right) = 0.$$

Denote by F_n the set of points $x \in E_n$ for which $\varphi(T; E_n; x)$ is even (see 2.1), and by \tilde{F}_n the set $E_n \setminus F_n$. Define the automorphism T_n by the equality

$$T_n(x) = \begin{cases} T(T'_{E_n})^{-1}T^2x, & \text{if } x \in T^{-2}T'_{E_n}F_n, \\ T(T'_{E_n})^{-1}Tx, & \text{if } x \in T^{-1}T'_{E_n}\tilde{F}_n, \\ Tx, & \text{if } x \in T^{-1}T'_{E_n}F_n, \\ T^2x, & \text{if } x \notin T^{-2}T'_{E_n}F_n \cup T^{-1}T'_{E_n}\tilde{F}_n \cup T^{-1}T'_{E_n}F_n. \end{cases}$$

It is clear that $T_n \in \mathfrak{A}_\tau$ and

$$d(T_n, T^2) < 2\mu E_n,$$

so that the limit of the sequence T_n is the automorphism T^2 , which does not belong to \mathfrak{A}_τ .

Now let $T_i \in \mathfrak{A}_\tau$ ($i = 1, 2, \dots$), and let the sequence $\{T_i\}_{i=1}^\infty$ be Cauchy. Choose from it a subsequence $\{T'_i\}_{i=1}^\infty$ for which

$$d(T'_n, T'_{n+1}) < \frac{1}{2^n}.$$

Let $T \in \mathfrak{A}_\tau$,

$$D_n = \{x \in M \mid \theta(T, T'_n; x) \neq \theta(T, T'_{n+1}; x)\},$$

and

$$E_n = \bigcup_{k=n+1}^\infty D_k.$$

Obviously,

$$E_n \supset E_{n+1}$$

and

$$\mu E_n < \frac{1}{2^n} \quad (n = 1, 2, \dots),$$

so that the set

$$E = \bigcap_{n=1}^\infty E_n$$

has measure 0. Put

$$g(x) = \theta(T, T'_n; x), \quad \text{if } x \notin E_n.$$

It is obvious that the mapping J , defined by the equality

$$Jx = T^{g(x)}x, \quad x \in M,$$

is an automorphism of the space M and $\tau_J \geq \tau$. This automorphism is the limit of the sequence $\{T'_i\}_{i=1}^\infty$ and of the original sequence $\{T_i\}_{i=1}^\infty$.

4.3. Fix an automorphism $R \in \mathfrak{A}_\tau$. For $T, S \in \mathfrak{A}_\tau$ put

$$G(T, S) = \{x \in M \mid \theta(R, T; x) \neq \theta(R, S; x)\},$$

$$H(T, S) = \{x \in M \mid \theta(T, R; x) \neq \theta(S, R; x)\}.$$

Define on \mathfrak{A}_τ the metric ρ_R by the equality

$$\rho_R(T, S) = \mu[G(T, S) \cup H(T, S)], \quad T, S \in \mathfrak{A}_\tau.$$

For any R the metric ρ_R is equivalent to the metric d . Indeed, if $T_i, T \in \mathfrak{A}_\tau$ and

$$T_i \xrightarrow{\rho_R} T,$$

then

$$T_i \xrightarrow{d} T,$$

which follows from the equality

$$G(T, T_i) = \{x \in M \mid \theta(T, T_i; x) \neq 1\}.$$

Conversely, let $T_i, T \in \mathfrak{A}_\tau$ and

$$T_i \xrightarrow{d} T.$$

Then

$$\mu G(T, T_i) \rightarrow 0.$$

It remains to show that

$$\mu H(T, T_i) \rightarrow 0.$$

Since

$$\mu[M \setminus G(T, T_i)] \rightarrow 1,$$

for almost every $x \in M$ there is an index i_0 such that for all $i > i_0$

$$\theta_k(R, T_i; x) = \theta_k(R, T; x), \quad 1 \leq k \leq \theta(T, R; x).$$

Then

$$\theta(T, R; x) = \theta(T_i, R; x) \quad \text{for } i > i_0.$$

It follows from this that

$$\mu H(T, T_i) \rightarrow 0.$$

4.4. Theorem. The space \mathfrak{A}_τ is complete in the metric ρ_R and is separable.

Proof. Let $\{T_i\}_{i=1}^\infty$ be an arbitrary fundamental sequence in the metric ρ_R , $T_i \in \mathfrak{A}_\tau$. Then there exists an automorphism $T \in \mathfrak{A}'_\tau$ such that $T_i \xrightarrow{d} T$ (Theorem 4.2). We shall show that $T \in \mathfrak{A}_\tau$. Choose from $\{T_i\}_{i=1}^\infty$ a subsequence $\{T'_i\}_{i=1}^\infty$ for which $\mu H(T'_i, T'_{i+1}) < 1/2^i$, and put

$$C_i = \bigcup_{k=i+1}^\infty H(T'_k, T'_{k+1}).$$

Obviously, $\mu C_i < 1/2^i$, so that $\mu(\bigcap_{i=1}^{\infty} C_i) = 0$. Consequently, the equality $h(x) = \theta(T'_i, R; x)$ for $x \notin C_i$ defines on M a function $h(\cdot)$ such that $Rx = T^{h(x)}x$ and $h(x) = \theta(T, R; x)$. Hence $\tau_T = \tau_R$ (see 3.1), and $T \in \mathfrak{A}_\tau$.

We now prove separability. The space \mathfrak{A}'_τ with the metric d_R (see 4.1) is homeomorphic to a subspace of the space \mathfrak{B} of all measurable integer-valued functions on M with the metric $\rho(f, g) = \mu\{x \in M \mid f(x) \neq g(x)\}$, $f, g \in \mathfrak{B}$ (to each automorphism $T \in \mathfrak{A}_\tau$ there corresponds the function $\theta(R, T; \cdot) \in \mathfrak{B}$). Consider \mathfrak{B} as a part of the space of all finite measurable functions on M with the metric

$$r(f, g) = \int \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} d\mu.$$

Convergence in this space is convergence in measure. It is clear that convergence in measure induces on \mathfrak{B} convergence in the metric ρ . Thus the space \mathfrak{A}'_τ is homeomorphic to a subspace of the space of all finite measurable functions with metric r , and that space, as is known, is separable. Hence the separability of \mathfrak{A}'_τ and \mathfrak{A}_τ follows.

4.5. Lemma. Let τ be such that the automorphisms from \mathfrak{A}_τ are aperiodic. Then the distance in \mathfrak{A}'_τ from any automorphism $T \in \mathfrak{A}_\tau$ to the set \mathfrak{P}_p of automorphisms from \mathfrak{A}_τ of period p is equal to $1/p$.

Proof. Let $\{\varepsilon_i\}_{i=0}^{\infty}$ be a sequence of positive numbers tending to 0, $\varepsilon_0 < 1$. Using the aperiodicity of T , find a set E_0 such that $E_0 \cap T^i E_0 = \Lambda$, $i = 1, \dots, p-1$, and

$$\mu \left(\bigcup_{i=0}^{p-1} T^i E_0 \right) > 1 - \varepsilon_0$$

(see [1]). Put $M_1 = M \setminus \bigcup_{i=0}^{p-1} T^i E_0$. Suppose the set M_n has been constructed. Find a set E_n such that $E_n \subset M_n$, $E_n \cap (T'_{M_n})^i E_n = \Lambda$, $i = 1, \dots, p-1$, and

$$\mu_{M_n} \left(\bigcup_{i=0}^{p-1} (T'_{M_n})^i E_n \right) > 1 - \varepsilon_n.$$

Put $M_{n+1} = M_n \setminus \bigcup_{i=0}^{p-1} (T'_{M_n})^i E_n$ and continue the process. Then define the automorphism P by the equality

$$Px = \begin{cases} Tx, & \text{if } x \in \bigcup_{i=0}^{p-2} T^i E_0, \\ T^{-p+1}x, & \text{if } x \in T^{p-1} E_0, \\ T'_{M_n} x, & \text{if } x \in \bigcup_{i=0}^{p-2} (T'_{M_n})^i E_n, \quad n = 1, 2, \dots, \\ (T'_{M_n})^{-p+1}x, & \text{if } x \in (T'_{M_n})^{p-1} E, \quad n = 1, 2, \dots. \end{cases}$$

It is clear that the automorphism P has period p , $d(T, P) \leq \frac{1}{p} + \varepsilon_0$, and $\tau_P \geq \tau$. Since ε_0 is arbitrarily small, $d(T, \mathfrak{P}_p) \leq 1/p$. The inequality $d(T, \mathfrak{P}_p) \geq 1/p$ is obvious.

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4.6. Lemma. Let τ be such that the automorphisms in \mathfrak{A}_τ are ergodic. If automorphisms P and Q from \mathfrak{A}'_τ have period p , then there exists an automorphism $U \in \mathfrak{A}_\tau$ such that $UP = QU$.

Proof. Let E and F be fundamental sets of the automorphisms P and Q , respectively, and let I be a mapping of E onto F compatible with T (see 2.3). Put

$$Ux = \begin{cases} Ix, & \text{if } x \in E, \\ Q^k I P^{-k} x, & \text{if } x \in P^k E, \quad k = 1, \dots, p-1. \end{cases}$$

It is clear that $U \in \mathfrak{A}'_\tau$. The equality $UP = QU$ is obvious.

4.7. Theorem. Let τ be such that the automorphisms in \mathfrak{A}_τ are ergodic. Then the set of automorphisms isomorphic to a given ergodic automorphism T_0 is everywhere dense in \mathfrak{A}_τ .

Proof. In the space \mathfrak{A}_τ find an automorphism T isomorphic to T_0 (according to Theorem 2.7), and show that the set of automorphisms UTU^{-1} is dense in \mathfrak{A}_τ . Fix $S \in \mathfrak{A}_\tau$, $\varepsilon > 0$, and such a p that $4/p < \varepsilon$. Find in \mathfrak{A}'_τ automorphisms P and Q of period p such that $d(T, P) < 2/p$, $d(Q, S) < 2/p$ (Lemma 4.5). Let $U \in \mathfrak{A}'_\tau$ be such an automorphism that $UP = QU$. Then

$$d(UTU^{-1}, S) < d(UTU^{-1}, UPU^{-1}) + d(UPU^{-1}, S) = d(T, P) + d(Q, S) < 4/p < \varepsilon.$$

It is easy to verify that, since $U \in \mathfrak{A}'_\tau$, we have $UTU^{-1} \in \mathfrak{A}_\tau$. The theorem is proved.

4.8. Theorem. Let τ be such that the automorphisms in \mathfrak{A}_τ are ergodic. Then the set of automorphisms with purely continuous spectrum is an everywhere dense G_δ in the space \mathfrak{A}_τ ; the set of automorphisms with zero entropy is an everywhere dense G_δ in the space \mathfrak{A}_τ ; the set of mixing transformations is a set of first category in the space \mathfrak{A}_τ .

Proof. From Theorem 2.7 it follows that these sets are nonempty. From Theorem 4.7 it follows that they are everywhere dense in \mathfrak{A}_τ . Since the set of automorphisms with purely continuous spectrum and the set of automorphisms with zero entropy are G_δ in the space \mathfrak{A}_d of all aperiodic automorphisms of the space M (see [1] and [3]), the same is true with respect to the space \mathfrak{A}_τ . The set of mixing transformations has first category in the space \mathfrak{A}_d (see [1]); its complement is an everywhere dense G_δ in \mathfrak{A}_d . Therefore the complement of the set of mixing transformations is an everywhere dense G_δ in \mathfrak{A}_τ , and the set itself has first category in \mathfrak{A}_τ .

4.9. Besides the uniform topology, a very popular topology on the set \mathfrak{A} of all automorphisms of the space M is the weak topology. A neighborhood of an automorphism $T_0 \in \mathfrak{A}$ in the weak topology is defined as the set of automorphisms T satisfying a finite number of inequalities of the form $\rho(T_0 A, T A) < \varepsilon$, where A is a measurable set and ε is a positive number (see [1]). The space \mathfrak{A} turns out to be metrizable: if D_1, D_2, \dots is a multiplicative basis of the space M , then the metric

$$\sigma(S, T) = \sum_{k=1}^{\infty} \frac{1}{2^k} \rho(S D_k, T D_k)$$

defines the weak topology on \mathfrak{A} . Since $\rho(S A, T A) \leq d(S, T)$ for any measurable set A , it follows that $\sigma(S, T) \leq d(S, T)$.

4.10. Theorem. Let τ be a partition such that the automorphisms of the set \mathfrak{A}_τ are ergodic. Then the set \mathfrak{A}_τ is dense in the space \mathfrak{A} in the weak topology.

Proof. Fix an automorphism $T \in \mathfrak{A}$ and $\varepsilon > 0$. By Theorem 4.7, in the set $\mathfrak{A}_{\tau T}$ there is an automorphism T_0 with discrete spectrum consisting of roots of unity of degree 2^k , $k = 1, 2, \dots$, for which $d(T_0, T) < \varepsilon/2$, and hence $\sigma(T_0, T) < \varepsilon/2$.

Since T_0 has discrete spectrum, there exists in L^2 a complete orthonormal system of functions, each element of which is an eigenfunction of the unitary operator U_{T_0} generated by it. Let f_k be the eigenfunction corresponding to a root of unity of degree 2^k ($k = 1, 2, \dots$). Since, for any $x \in M$,

$$f_k(T_0^{2^k} x) = f_k(x),$$

while

$$f_k(T_0 x) \neq f_k(x),$$

the space M is divided into 2^k sets $B_k^1, \dots, B_k^{2^k}$ of equal measure, on each of which the function f_k assumes a constant value, and, under the corresponding

numbering,

$$T_0 B_k^i = B_k^{i+1} \quad (i = 1, \dots, 2^k - 1), \quad T_0 B_k^{2^k} = B_k^1.$$

We shall call these sets sets of rank k . Moreover,

$$B_k^1 \supset B_{k+1}^1$$

(for the corresponding numbering of the sets of rank $k + 1$). Since the system of eigenfunctions is complete in L^2 , the system of sets $\{B_k^i\}_{i,k}$ forms a basis of the space M .

Let $S \in \mathfrak{A}_\tau$, and let I_{ni} be a mapping of the set B_n^i onto the set B_n^{i+1} ($i = 1, \dots, 2^n - 1$; $n = 1, 2, \dots$), compatible with S (see §2.3). Define the automorphism S_n by

$$S_n x = \begin{cases} I_{ni} x, & \text{if } x \in B_n^i, \quad i = 1, 2, \dots, 2^n - 1, \\ S'_{B_n^1} I_{n1}^{-1} \dots I_{n2^n-1}^{-1} x, & \text{if } x \in B_n^{2^n}. \end{cases}$$

It is clear that $S_n \in \mathfrak{A}_\tau$ (Lemma 2.2) and

$$S_n B_k^i = T_0 B_k^i$$

for $n \geq k$. We show that $S_n \rightarrow T$ in the weak topology. Let A be a measurable set, δ a positive number, and B a set consisting of a finite number of sets of the system $\{B_k^i\}$ such that

$$\rho(A, B) < \delta/2.$$

If

$$N = \max\{k \mid B_k^i \subset B\},$$

then $S_n B = T_0 B$ for all $n > N$, and therefore

$$\rho(T_0 A, S_n A) \leq \rho(T_0 A, T_0 B) + \rho(T_0 B, S_n B) + \rho(S_n B, S_n A) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Thus there exists a number N_0 such that for all $n > N_0$ we have

$$\sigma(S_n, T_0) < \varepsilon/2,$$

and then

$$\sigma(T, S_n) < \sigma(T, T_0) + \sigma(T_0, S_n) < \varepsilon.$$

4.11. If in the preceding item, instead of the automorphism $T \in \mathfrak{A}$, one takes an automorphism $S \in \mathfrak{A}_\tau$, then we see that in \mathfrak{A}_τ the weak topology does not coincide with the uniform topology: the sequence S_n converges to $S \in \mathfrak{A}_\tau$ weakly, but does not converge in the uniform topology. The weak topology is apparently useless for the study of the space \mathfrak{A}_τ .

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Received by the Editors

July 14, 1967

REFERENCES

1. Rokhlin V. A., Selected questions of the metric theory of dynamical systems, *UMN* **4**, no. 2 (1949), 57-128.
2. Rokhlin V. A., On the fundamental concepts of measure theory, *Matem. sb.* **25** (1949), 107-150.
3. Rokhlin V. A., On the entropy of a metric automorphism, *DAN SSSR* **124**, no. 5 (1959), 980-983.
4. Rokhlin V. A., Lectures on the entropy theory of transformations with invariant measure, *UMN* **22**, no. 5 (1967), 3-56.
5. Abramov L. M., Entropy of a derived automorphism, *DAN SSSR* **128**, no. 4 (1959), 647-650.
6. Vershik A. M., A theorem on lacunary isomorphism of monotone sequences of partitions, *Funkts. analiz* **2**, no. 3 (1968), 17-21.

Note: Figure translations are in progress. See original paper for figures.

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