

# ON THE BUBNOV- GALERKIN METHOD FOR PARABOLIC EQUATIONS IN A HILBERT SPACE

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON THE BUBNOV-GALERKIN METHOD FOR PARABOLIC EQUATIONS IN A HILBERT SPACE

*(Presented by Academician A. Yu. Ishlinskii, March 27, 1967)*

1. In a Hilbert space  $H$  we consider the problem

$$v' + A(t)v = f(t, v) \quad (0 \leq t \leq T), \quad v(0) = v_0. \quad (1)$$

Its generalized solutions were studied in <sup>(1)</sup>. An **approximate solution** of problem (1) is the generalized solution of the problem

$$v'_n + P_n A(t)v_n = P_n f(t, v_n) \quad (0 \leq t \leq T), \quad v_n(0) = P_n v_0. \quad (2)$$

Here  $P_n$  is the operator of orthogonal projection onto the linear span  $H_n$  of the first  $n$  elements of the system  $\{e_i\}$ .

It is assumed that  $A(t)$ , for each  $t \in [0, T]$ , is a positive definite self-adjoint operator whose domain  $D$  does not depend on  $t$ . It is assumed that the following hold: (a)  $A(t)A^{-1}(0)$  has only discontinuities of the first kind; (b)  $A(t)A^{-1}(0)$  has bounded variation; (c)  $A(t)A^{-1}(0)$  satisfies a Hölder condition with exponent  $\varepsilon > 0$ .

It is assumed that all  $e_i \in D$  and (in the convergence theorems) that the system  $\{e_i\}$  is complete in the energy norm, i.e., that the set

$$\bigcup_{n=1}^{\infty} A^{1/2}(0)H_n$$

is dense in  $H$ .

Below we give results on problem (2), obtained in 1957-1958 <sup>(2)</sup>.

2. Since  $P_n A(t)$  is a positive definite operator in  $H_n$  and  $H_n \subset D$ , for any  $0 \leq \alpha \leq 1$ ,  $0 \leq t \leq T$ ,  $n = 1, 2, \dots$ , the functions

$$\varphi_\alpha = \varphi_\alpha(n, t) = \|A^\alpha(t)[P_n A(t)]^{-\alpha} P_n\|,$$

$$\psi_\alpha = \psi_\alpha(n, t) = \|[P_n A(t)]^\alpha P_n A^{-\alpha}(t)\|$$

are defined.

**Lemma 1.** The functions  $\varphi_\alpha$  and  $\psi_\alpha$  are logarithmically convex for  $0 \leq \alpha \leq 1$ ;  $\varphi_\alpha \leq 1$  for  $0 \leq \alpha \leq 1/2$ . If  $\varphi_1 \leq C_1$ , then  $\varphi_\alpha \leq C_1^{2\alpha-1}$  for  $1/2 \leq \alpha \leq 1$ . If  $\psi_{\alpha_0} \leq D_{\alpha_0}$  for some  $\alpha_0$  from  $[0, 1]$ , then  $\psi_\alpha \leq D_{\alpha_0}^{\alpha/\alpha_0}$  for all  $\alpha$  from  $[0, \alpha_0]$ .

The proof is based on Heinz inequalities (see <sup>(3,4)</sup> and <sup>(5)</sup>, p. 205).

The estimate  $\varphi_{1/2} = 1$  was used in <sup>(6)</sup>. The estimate  $\varphi_1 \leq C_1$  holds if there exists an invertible operator  $B$  such that  $D(B) = D$ ,

$$C_1(A(t)v, Bv) \geq \|A(t)v\| \cdot \|Bv\| \quad \text{for } v \in D,$$

and  $H_n$  are invariant with respect to  $B$  <sup>(7)</sup>. If  $B^{-1}$  is bounded, then the estimate  $\varphi_1 \leq C_1$  means that for the systems  $\{e_i\}$  and  $\{AB^{-1}e_i\}$  the condition (A) of N. I. Pol'skii is satisfied (see <sup>(10,8)</sup>, p. 124). The estimate  $\psi_{\alpha_0} \leq D_{\alpha_0}$  in the case  $\alpha_0 = 1/2, 1$  holds if there exists such a bounded invertible operator  $B_{\alpha_0}$  that

$$D(B_{\alpha_0}) = D(A^{\alpha_0}(0)) = D(A^{\alpha_0}(t)),$$

and  $H_n$  are invariant with respect to  $B_{\alpha_0}$ .

3. Denote by  $U_n(t, s)$  ( $0 \leq s \leq t \leq T$ ) the operator such that the function  $v_n = U_n(t, s)v_{0n}$  is the solution of the homogeneous problem

$$v'_n + P_n A(t)v_n = 0 \quad (s \leq t \leq T), \quad v_n(s) = v_{0n} \in H_n. \quad (3)$$

With the aid of the methods developed in [1] and the estimates of item 2, the following is established:

**Lemma 2.** Suppose that (a) is satisfied and  $\varphi_1 \leq C$ . Then for every  $v \in H_n$  the estimates

$$\begin{aligned} \max \left\{ \|[P_{nA}(0)]^{1/2} U_n(t, s)v\|, \left( \int_0^t \|P_{nA}(\tau) U_n(\tau, s)v\|^2 d\tau \right)^{1/2} \right\} \leq \\ \leq K \|[P_{nA}(0)]^{1/2} v\| \end{aligned} \quad (4)$$

are valid.

**Lemma 3.** The estimates

$$\| [P_{nA}(t)]^\beta U_n(t, s) [P_{nA}(s)]^{-\alpha} P_n \| \leq K |t - s|^{\alpha - \beta} \quad (5)$$

hold for  $0 \leq \alpha \leq \beta \leq 1/2$ , if (b) is satisfied; for  $0 \leq \alpha \leq \beta < 1/2 + \varepsilon$ , if (c) is satisfied; for  $0 \leq \alpha \leq 1$ ,  $0 \leq \beta < 1 + \varepsilon$ ,  $\alpha \leq \beta$ , if (c) is satisfied and  $\varphi_1 \leq C_1$ .

4. The uniform (in  $n$  and  $t$ ) estimates of items 2 and 3 make it possible to establish the convergence of the solutions of problem (3) (for  $v_{0n} = P_{nv}0$ ) to the solution  $v(t) = U(t, s)v_0$  of the problem

$$v' + A(t)v = 0 \quad (s \leq t \leq T), \quad v(s) = v_0. \quad (6)$$

First of all, the limiting relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} \| U_n(t, s)v_m - U(t, s)v_m \| = 0 \quad (7)$$

is established. For this purpose it is first shown that  $U_n(t, s)x_m$  converges weakly to  $U(t, s)x_m$ . The proof of this fact is based on the inequalities

$$\| A^{1/2}(0)U_n(t, s)v_m \| \leq K \| A^{1/2}(0)v_m \|,$$

$$\| [U_n(t + \Delta t, s + \Delta s) - U(t, s)]v_m \| \leq [|\Delta t|^{1/2} + |\Delta s|^{1/2}] K \| A^{1/2}(0)v_m \|, \quad (8)$$

which follow from Lemmas 1-3. Then the convergence of  $\| U_n(t, s)x_m \|$  to  $\| U(t, s)x_m \|$  is proved. For this one uses the weak convergence of  $\tilde{U}_n(2t - s, s)v_m$  to  $\tilde{U}(2t - s, s)v_m$ , constructed from the operator

$$\tilde{A}(\tau) = \begin{cases} A(\tau), & s \leq \tau \leq t, \\ A(2t - \tau), & t \leq \tau \leq 2t, \end{cases} \quad (9)$$

and the formulas  $\tilde{U}_n(2t - s, s) = U_n^*(t, s)U_n(t, s)$ ,  $\tilde{U}(2t - s, s) = U^*(t, s)U(t, s)$ . Further, by means of (7), Lemmas 1-3, and the moment inequality (9), the following are established:

**Theorem 1.** For every  $v_0 \in H$  the limiting relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} |t - s|^\alpha \| A^\alpha(0)[U_n(t, s)P_{nv}0 - U(t, s)v_0] \| = 0 \quad (10)$$

holds for  $\alpha = 0$ , if (a) is satisfied and  $\varphi_1 \leq C_1$ ; for  $0 \leq \alpha < 1/2$ , if (b) is satisfied; for  $0 \leq \alpha \leq 1/2$ , if (c) is satisfied; for  $0 \leq \alpha < 1$ , if (b) or (c) is satisfied and  $\varphi_1 \leq C_1$ .

In applications to partial differential equations, (10) means convergence of the solutions together with their derivatives with respect to the spatial variables.

**Theorem 2.** For every  $v_0 \in H$  the limiting relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} |t - s| \left\| \frac{\partial}{\partial t} U_n(t, s) P_n v_0 - \frac{\partial}{\partial t} U(t, s) v_0 \right\| = 0, \quad (11)$$

holds, if (c) is satisfied and  $\varphi_1 \leq C_1$ .

If  $v_0 \in D(A^\gamma(0))$ , then the preceding assertions can be sharpened.

**Theorem 3.** For every  $v_0 \in D(A^{1/2}(0))$  the limit relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} \|A^\alpha(0) [U_n(t, s) P_n v_0 - U(t, s) v_0]\| = 0, \quad (12)$$

holds, if (a) is satisfied,  $\varphi_1 \leq C_1$  and  $\psi_{1/2} \leq D_{1/2}$ .

**Theorem 4.** For every  $v_0 \in D(A^\gamma(0))$  the limit relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} |t - s|^{\alpha - \gamma} \|A^\alpha(0) [U_n(t, s) P_n v_0 - U(t, s) v_0]\| = 0 \quad (13)$$

holds for  $0 \leq \gamma \leq \alpha < 1$ , if (b) or (c) is satisfied,  $\varphi_1 \leq C_1$  and  $\psi_\gamma \leq D_\gamma$ .

**Theorem 5.** For every  $v_0 \in D(A^\gamma(0))$  the limit relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} |t - s|^{1 - \gamma} \left\| \frac{\partial}{\partial t} U_n(t, s) P_n v_0 - \frac{\partial}{\partial t} U(t, s) v_0 \right\| = 0 \quad (14)$$

holds for  $0 \leq \gamma \leq 1$ , if (c) is satisfied,  $\varphi_1 \leq C_1$  and  $\psi_\gamma \leq D_\gamma$ .

Finally, we formulate conditions under which not only the solutions  $v_n$  of problems (3) (with  $v_{0n} = P_n v_0$ ) converge to the solution  $v$  of problem (6), and not only the derivatives  $v'_n \rightarrow v'$  (Theorems 2 and 5), but also the residuals in equation (6) tend to zero.

**Theorem 6.** For every  $v_0 \in D(A^\gamma(0))$  the limit relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} |t - s|^{1 - \gamma} \|A(t) [U_n(t, s) P_n v_0 - U(t, s) v_0]\| = 0 \quad (15)$$

holds for  $0 \leq \gamma \leq 1$ , if (c) is satisfied,  $\varphi_1 \leq C_1$ , and there exists an operator  $B$  having an inverse such that  $D(B) = D$  and the  $H_n$  are invariant with respect to  $B$ .

Recall (Section 2) that  $\varphi_1 \leq C_1$ , if  $C_1(A(t)v, Bv) \geq \|A(t)v\| \cdot \|Bv\|$  for  $v \in D$ .

5. Since (see (1)) the solution of problem (2) is a solution of the equation

$$v_n(t) = U_n(t, 0)P_n v_0 + \int_0^t U_n(t, s)P_n f[s, v_n(s)] ds, \quad (16)$$

the assertions given in Section 4 make it possible to study the convergence of the solutions of problem (6) to the solution of problem (1).

We first consider the case of a linear equation. We shall state only the theorem on the convergence of the residuals to zero.

**Theorem 7.** Suppose that the conditions of Theorem 6 are satisfied. Then the limit relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} \left\| (I - P_n)A \int_0^t U_n(t, s)P_n f(s) ds \right\| = 0 \quad (17)$$

holds for any function  $f(t)$  satisfying the Hölder condition.

We now proceed to the consideration of the nonlinear equation (16).

**Theorem 8.** There exists a segment  $[0, h] \subset [0, T]$ , independent of  $n$ , such that equation (16) has at least one solution  $v_n(t)$ , defined on  $[0, h]$ . The family  $A^\alpha(0)v_n(t)$  is compact in  $C([0, T], H)$ . Its limit points have the form  $A^\alpha(0)v(t)$ , where  $v(t)$  are generalized solutions of problem (1). If problem (1) has a unique solution  $v(t)$ , then

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq h} \|A^\alpha(0)[v_n(t) - v(t)]\| = 0. \quad (18)$$

These assertions hold for some  $\alpha \in [0, 1/2)$ , if (a) is satisfied, the operator  $A^{-1}(0)$  is completely continuous, the operator  $f(t, A^{-\alpha}(0)w(t))$  acts boundedly and continuously from  $C([0, T], H)$  into  $B_2([0, T], H)$ ,  $v_0 \in D(A^{1/2}(0))$ ,  $\varphi_1 \leq C_1$ , and  $\psi_{1/2} \leq D_{1/2}$ ; and for some  $\alpha \in [0, 1)$ , if (b) or (c) is satisfied, the operator  $A^{-1}(0)$  is completely continuous, the operator  $f(t, A^{-\alpha}(0)w(t))$  acts boundedly and continuously from  $C([0, T], H)$  into  $B_p([0, T], H)$  for some  $p > 1/(1 - \alpha)$ ,  $v_0 \in D(A^\gamma(0))$  for some  $\alpha < \gamma \leq 1$ ,  $\varphi_1 \leq C_1$ , and, if  $\alpha \geq 1/2$ ,  $\psi_\gamma \leq C_\gamma$ .

Finally, if the hypotheses of Theorem 6 are satisfied and the nonlinear operator  $f(t, v)$  is sufficiently smooth, then the residuals converge to zero (uniformly in  $t$ ).

6. In the preceding sections, the convergence, uniform in  $t$ , of the solutions of problem (2) to the solution of problem (1) was investigated. The use of the estimates from (1) makes it possible to study convergence in the space  $B_2([0, T], H)$ , and in some cases also in an arbitrary space  $B_p([0, T], H)$ .

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