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MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 519.48

MATHEMATICS

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ON THE STRUCTURE OF THE FIELD OF FRACTIONS OF THE ENVELOPING ALGEBRA OF A SEMISIMPLE LIE ALGEBRA

1. Let $\mathfrak{g}$ be a semisimple Lie algebra, $U(\mathfrak{g})$ its enveloping algebra, and $D(\mathfrak{g})$ the field of fractions of the algebra $U(\mathfrak{g})$. In ([1]) the authors put forward a hypothesis on the structure of the field $D(\mathfrak{g})$ for any algebraic Lie algebra. This hypothesis was proved in ([1]) for nilpotent Lie algebras, for the full matrix algebra, and for the algebra of all matrices with zero trace. In the present note we study the case of an arbitrary complex semisimple Lie algebra.*

Our approach to the study of the structure of $D(\mathfrak{g})$ is suggested by the theory of infinite-dimensional representations of complex semisimple Lie groups. It is based on realizing the algebra $U(\mathfrak{g})$ as a ring of differential operators on the basic affine space (see ([2])) and on extending this algebra by means of virtual Laplace operators introduced by us.** The main result consists in a description of the structure of the extended enveloping algebra $\tilde{U}(\mathfrak{g})$ (Theorem 1) and of its field of fractions $\tilde{D}(\mathfrak{g})$ (Theorem 2).

2. We introduce notation: $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$; Δ (Δ_+, Δ_-) is the set of roots (positive roots, negative roots) of the algebra $\mathfrak{g}$; X_α is the root vector corresponding to the root $\alpha \in \Delta$; $\mathfrak{n}_+$ $(\mathfrak{n}_-)$ is the subalgebra generated by all X_α, $\alpha \in \Delta_+$ (Δ_-). Let G be a simply connected Lie group for which $\mathfrak{g}$ is the Lie algebra. We denote by H, N_+, N_- the subgroups corresponding to the subalgebras $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-$. The homogeneous space $A = N_-\backslash G$ of right cosets in G modulo N_- will be called the basic affine space. Since the subgroup H normalizes N_-, it acts on A by left translations: an element $h \in H$ sends the coset N_-g to N_-hg. As is known ([3]), the subset N_-HN_+ is open and dense in G. Therefore a certain open dense subset in A is naturally identified with the subgroup HN_+. On this subset we define coordinates τ_i, $i = 1, 2, \ldots, k = \dim H$; t_j, $j = 1, 2, \ldots, n = \dim N_+$, as follows:

$$\tau_i(hn_+) = \tau_i(h) = e^{\langle \sigma_i, \ln h \rangle};$$

$t_j(hn_+) = t_j(n_+)$ is equal to the coefficient of X_{α_j} in the expansion of $\ln n_+$. Here ln denotes some preimage of an element of the Lie group in the corresponding Lie algebra with respect to the canonical mapping; $\sigma_1, \dots, \sigma_k$ are the highest weights of the elementary representations of the group G; $\alpha_1, \dots, \alpha_n$ are the positive roots of the algebra $\mathfrak{g}$.

The variety A is naturally endowed with the structure of an affine algebraic variety. Namely, the ring F of regular functions on A, by definition, consists of the functions f having the following property: the linear span of the functions obtained from f by all transformations from the group G has finite dimension.

* In fact, the method we use is applicable to any split reductive algebra over a field of characteristic zero.

** The use of these operators is analogous to the introduction of Bott generators in the study of the cohomology ring of the classifying space BG. Just as in topology, the introduction of virtual Laplace operators simplifies many formulas of representation theory.

Lemma 1. The ring F is contained in $C[\tau_1, \dots, \tau_k, t_1, \dots, t_n]$, and its quotient field coincides with $(\tau_1, \dots, \tau_k, t_1, \dots, t_n)$.

2. The main object of our investigation will be the ring R of differential operators on A that preserve the ring F. In the ring R the following additional structures are introduced:

1) The filtration
$$F = R_0 \subset R_1 \subset \cdots \subset R_m \subset \cdots \subset R,$$
associated with the order of a differential operator.

2) The grading
$$R = \sum_{\lambda} {}^{\lambda}R,$$
associated with the action of left translations.

To each integral k-dimensional vector $\lambda = (\lambda_1, \dots, \lambda_k)$ there corresponds the subspace ${}^{\lambda}R$ of those operators which, under left translation by $h \in H$, are multiplied by
$$\tau^{\lambda}(h) = \prod_{i=1}^{k} \tau_i^{\lambda_i}(h).$$

3) The decomposition into a direct sum
$$R = \sum R^{\mu},$$
associated with the action of right translations. To each k-dimensional vector $\mu = (\mu_1, \dots, \mu_k)$ with integral nonnegative coordinates there corresponds the subspace R^{μ}, in which the representation G is realized with multiplicity equal to that of the irreducible representation T_{μ} with highest weight μ (we identify the integral vector μ with the linear functional $\mu_1\sigma_1 + \cdots + \mu_k\sigma_k$ on $\mathfrak{h}$).

By the letter R with several indices we denote the intersection of the subsets R corresponding to each of these indices; thus, ${}^{\lambda}R^{\mu}_{m}$ means

$$R_m \cap {}^{\lambda}R \cap R^{\mu}.$$

In what follows a special role is played by the ring ${}^{0}R^{0}$, which we call the ring of virtual Laplace operators and denote by $\widetilde{Z}(\mathfrak{g})$. To each element $X \in \mathfrak{h}$ there corresponds the infinitesimal operator of left translation L_X:

$$L_X f(N\_g) = \frac{d}{dt} f(N\_ \exp(tX)g)\Big|_{t=0},$$

which evidently belongs to ${}^{0}R^{0}_{1}$. We denote by ψ the homomorphism of the algebra $U(\mathfrak{h})$ into $\widetilde{Z}(\mathfrak{g})$ that takes $X \in \mathfrak{h}$ to $L_X + \langle \rho, X\rangle$, where

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha = \sum_{i=1}^{k} \sigma_i.$$

In particular, if $\alpha_1^*, \ldots, \alpha_k^*$ is the basis in $\mathfrak{h}$ dual to the basis $\sigma_1, \ldots, \sigma_k$, then

$$\psi(\alpha_i^*) = \tau_i \partial/\partial\tau_i + 1.$$

Let $\varphi : U(\mathfrak{g}) \to R$ be the natural representation of the enveloping algebra in the ring of differential operators on A, corresponding to the action of the group G on A by right translations. It is clear that the image of the algebra $U(\mathfrak{g})$ is contained in ${}^{0}R$, and the image of its center $Z(\mathfrak{g})$ is contained in $\widetilde{Z}(\mathfrak{g})$.

The subring in ${}^{0}R$ generated by $\varphi(U(\mathfrak{g}))$ and $\widetilde{Z}(\mathfrak{g})$ we call the extended enveloping algebra and denote by $\widetilde{U}(\mathfrak{g})$.* In the algebra $\widetilde{U}(\mathfrak{g})$ one can define the action of the Weyl group W in the following way. Recall (see (*), Chapter III, § 5) that to each element $s \in W$ there corresponds a Weyl operator B_s, acting in a certain space of smooth rapidly decreasing functions on A by the formula

$$B_s f(N\_g) = \int_{N\_\cap s^{-1}N_+ s} f(N\_ s^{-1} n g)\, dn.$$

It turns out that for any operator $L \in \widetilde{U}(\mathfrak{g})$ the operator $L^s = B_{sLB_s^{-1}}$ also belongs to $\widetilde{U}(\mathfrak{g})$. We denote by $\widetilde{U}(\mathfrak{g})^W$ the set of operators $L \in \widetilde{U}(\mathfrak{g})$ having the property

$$L^s = L$$

for all $s \in W$.

Theorem 1. The mapping φ establishes an isomorphism of $U(\mathfrak{g})$ and $\widetilde{U}(\mathfrak{g})^W$. The algebra $\widetilde{U}(\mathfrak{g})$ is naturally isomorphic to the tensor product

$$U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \widetilde{Z}(\mathfrak{g}).$$

The mapping ψ is an isomorphism of W-modules $U(\mathfrak{h})$ and $\widetilde{Z}(\mathfrak{g})$.

We now pass to the quotient field $\widetilde{D}(\mathfrak{g})$ of the algebra $\widetilde{U}(\mathfrak{g})$. Since the algebra $\widetilde{U}(\mathfrak{g})$ is embedded in R, the field $\widetilde{D}(\mathfrak{g})$ is embedded in the quotient field D of the algebra R,

* Apparently, $\widetilde{U}(\mathfrak{g})$ in fact coincides with ${}^0R$, but we have not yet succeeded in proving this.

which, by virtue of Lemma 1, is isomorphic to the standard field $D_{n+k,0}$ generated by the operators $\tau_i,\ \partial/\partial\tau_i,\ 1 \le i \le k,\ t_j,\ \partial/\partial t_j,\ 1 \le j \le n$. Let ${}^0D$ be the set of elements of D commuting with the left shifts. It is clear that ${}^0D$ is generated by the operators $\tau_i\partial/\partial\tau_i,\ 1 \le i \le k,\ t_j,\ \partial/\partial t_j,\ 1 \le j \le n$, and hence is isomorphic to the standard field $D_{n,k}$.

Theorem 2. *The field $\bar{D}(\mathfrak{g})$ coincides with ${}^0D$.*

The proof of Theorem 2 is based on the study of the structure of the spaces ${}^\lambda R^\mu$ as $\bar{Z}(\mathfrak{g})$-modules. This is done with the help of the Frobenius-Cartan duality for induced representations.

The resulting statement can be formulated as follows.

Theorem 3. *Let K be the field of fractions of the ring $Z(\mathfrak{g})$. The space*

$$ {}^\lambda R^\mu \otimes_{\bar{Z}(\mathfrak{g})} K $$

has dimension $d_\mu \cdot m_\mu(\lambda)$, where d_μ is the dimension of the irreducible representation T_μ, and $m_\mu(\lambda)$ is the multiplicity of the weight λ in the representation space T_μ.

Let us briefly indicate the scheme for deriving Theorem 2 from Theorem 3. First of all, one can verify that every element $x \in {}^0D$ can be written in the form $x = ab^{-1}$, where a and b belong to ${}^\lambda R$ for some λ. Replacing a and b by ac and bc, where $c \in {}^{-\lambda}R$, we obtain that ${}^0D$ coincides with the field of fractions of the ring ${}^0R$. Notice that here we use the nonemptiness of ${}^{-\lambda}R$, which follows from Theorem 3. It remains to verify that the rings ${}^0R$ and $U(\mathfrak{g})$ have one and the same field of fractions. For this it suffices to prove that the vector spaces

$$ {}^0R \otimes_{\bar{Z}(\mathfrak{g})} K \quad \text{and} \quad U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} K $$

coincide. Since the second space is contained in the first, it is only necessary to compare the dimensions of the intersections of these infinite-dimensional spaces with each of the finite-dimensional spaces

$$ {}^0R^\mu \otimes_{Z(\mathfrak{g})} K. $$

The dimension of one of the intersections is given by Theorem 3. The dimension of the second is computed similarly and turns out to be the same.*

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Received
11 III 1968

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* The fact that this dimension is equal to $d_\mu \cdot m_\mu(0)$ also follows from Kostant' s work (5).

Note: Figure translations are in progress. See original paper for figures.