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Abstract

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TEST SPACES FOR COHOMOLOGICAL DIMENSIONS OF PARACOMPACT SPACES

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Definition 1. A compactum P_n is called a **test space** for the cohomological dimension $\dim_G$ of paracompact spaces if, for every paracompact space X whose dimension $\dim X \leq n$, the equality holds

$$\dim_G X = \dim(X \times P_n) - \dim P_n.$$

Test spaces make it possible, in a number of cases, to reduce the proof of one or another assertion on cohomological dimension to the analogous assertion on the dimension dim. The existence of test spaces for the cohomological dimension of compact spaces with respect to an arbitrary abelian coefficient group was established in papers ([5],[7]). In this note test spaces are indicated for the cohomological dimensions of paracompact spaces.

Definition 2. The **cohomological dimension** $\dim_G X$ of a paracompact space X is the greatest integer n for which there exists a closed subset $A \subset X$ such that $H^n(X, A; G) \neq 0$. In this definition spectral cohomology groups based on the system of all open coverings are used.

Denote by $\widetilde{X}$ the space obtained from the space X by contracting the closed set A to a single point x_0. If $p : X \to \widetilde{X}$ is the identification map, then the open sets in $\widetilde{X}$ are taken to be precisely those sets whose full inverse image under p is open in X. If X is a paracompact space, then $\widetilde{X}$ will be a paracompact space, $\dim \widetilde{X} \leq \dim X$, and for any abelian coefficient group G the map p induces an isomorphism $H^*(X, A; G) \cong H^*(\widetilde{X}, x_0; G)$ (for the proof of the isomorphism see ([1]), pp. 5.10 and 4.10).

Lemma 1. *Let (X, A) be a pair of paracompact spaces; (Y, B) a pair of compact spaces; $\dim X = n$, $\dim Y = m$. Let the nonnegative integers s and t satisfy the inequality $s + t + m \geq t + n$. If $H^p(X, A; H^q(Y, B)) = 0$ for $p > s$ and $q > t$, then $H^i(X \times Y, (X \times B) \cup (A \times Y)) = 0$ for $i > s + m$. If, moreover, $H^s(X, A; H^m(Y, B)) \neq 0$ and $s > 0$, then $H^i(X \times Y, (X \times B) \cup (A \times Y)) \neq 0$ for $i = s + m$.*

Proof. There is an isomorphism

$$H^i(X \times Y, (X \times B) \cup (A \times Y)) \cong H^i(\widetilde{X} \times \widetilde{Y}, \widetilde{X} \vee \widetilde{Y}),$$

where

$$\widetilde{X} \vee \widetilde{Y} = (\widetilde{X} \times y_0) \cup (x_0 \times \widetilde{Y}).$$

Consider the exact sequence of the pair $(\widetilde{X} \times \widetilde{Y}, \widetilde{X} \vee \widetilde{Y})$

$$\cdots \to H^p(\widetilde{X} \times \widetilde{Y}, \widetilde{X} \vee \widetilde{Y}) \xrightarrow{j} H^p(\widetilde{X} \times \widetilde{Y}) \xrightarrow{i} H^p(\widetilde{X} \vee \widetilde{Y}) \to \cdots.$$

The homomorphism i is an epimorphism; therefore j is an isomorphism for $p > \max(\dim \widetilde{X}, \dim \widetilde{Y})$. Consequently, for $i > s + m$, or for $i = s + m$ and $s > 0$, the group $H^i(X \times Y, (X \times B) \cup (A \times Y))$ is isomorphic to the group $H^i(\widetilde{X} \times \widetilde{Y})$.

To compute the group $H^i(\widetilde{X} \times \widetilde{Y})$, consider the spectral sequence of the projection map $p_1 : \widetilde{X} \times \widetilde{Y} \to \widetilde{X}$ ([1]), p. 417). The term $E_2^{p,q}$ of this sequence is isomorphic to the group $H^p(\widetilde{X}, H^q(Y))^2$, and the term

E_∞ is associated with the group $H^*(\hat{X} \times \hat{Y})$. The sheaf $\underline{H^q(\hat{Y})}$ is constant. In fact, the projection $p_2 : \hat{X} \times \hat{Y} \to \hat{Y}$ assigns a homomorphism of the constant sheaf $H^q(\hat{Y})$ to the sheaf $\underline{H^q(\hat{Y})}$. This homomorphism, by Theorem 4.17.1 of ([1]), will be an isomorphism if the inverse images $p_1^{-1}(U)$ of neighborhoods of any point $x \in \hat{X}$ form a fundamental system of neighborhoods of the set $p_1^{-1}(x)$. But this condition is satisfied by virtue of the compactness of $\hat{Y}$. Consequently,

$$E_2^{p,q} \simeq H^p(\hat{X}, H^q(\hat{Y})) = 0 \quad \text{for } p + q > s + m.$$

If, moreover, $H^s(\hat{X}, H^m(\hat{Y})) \neq 0$, then $E_2^{s,m} \neq 0$, while $E_2^{p,q} = 0$ for $q > m$. Therefore $H^j(\hat{X} \times \hat{Y}) = 0$ for $j > s + m$, and $H^{s+m}(\hat{X} \times \hat{Y}) \neq 0$. The lemma is proved.*

Definition 3. A system $\{U_\alpha\}$ of open subsets of a space X is called a **large base** if, for every closed subset A of X and every neighborhood W of it, there exists a locally finite in X covering of the set A by elements of the system $\{U_\alpha\}$ contained in W.

Lemma 2. *If $\{U_\alpha\}$ is a large base of a paracompact space X and $\dim_G X = s$, then for at least one element U_α the group $H^s(X, X \setminus U_\alpha; G)$ is nonzero.*

Proof. In the space X there exists an open set U such that

$$H^s(X, X \setminus U; G) \neq 0.$$

Let a be a cocycle in the Godement resolution C^* of the constant sheaf G, representing a nonzero element of the group $H^s(X, X \setminus U; G)$. The support $S(a)$ of the cocycle a is contained in U.

Let $\{U_\gamma\}$ be a locally finite in X covering of the set $S(a)$ by elements of the large base $\{U_\alpha\}$ contained in U. The system $\{U_\gamma, X \setminus S(a)\}$ forms a locally finite covering of the space X. According to Godement ([1], p. 414), the sheaf of s-dimensional cocycles of the resolution C^* is a soft sheaf. By another theorem of Godement ([1], p. 3.6), there exists a decomposition of a section in a soft sheaf subordinate to a locally finite covering; moreover, from the proof of Godement' s theorem it follows that the summand in the decomposition of the cocycle a subordinate to the set $X \setminus S(a)$ may be taken to be zero. Let therefore

$$a = \sum a_\gamma$$

and $S(a_\gamma) \subset U_\gamma$. If all the groups $H^s(X, X \setminus U_\gamma; G)$ are zero, then for every γ there is a section b_γ in the sheaf C^{s-1} such that $S(b_\gamma) \subset U_\gamma$ and

$$\partial b_\gamma = a_\gamma.$$

Since the system $\{U_\gamma\}$ is locally finite, the section

$$b = \sum b_\gamma$$

is continuous and

$$\partial b = a.$$

This contradicts the nontriviality of the element of the cohomology group $H^s(X, X \setminus U)$ corresponding to the cocycle a. The lemma is proved.

We note that the well-known theorem of K. A. Sitnikov on obstruction follows from Lemma 2 with the aid of the duality law.

Lemma 3. *Let the group* $G = \sum_\alpha G_\alpha$, *where each group* G_α *is isomorphic to the group* Z_p, p *a fixed prime number. Then* $\dim_G X = \dim_{Z_p} X$ *for every paracompact space* X.

This lemma is a special case of the following theorem, proved for compacta by A. Grothendieck ([1], p. 88), and for paracompact spaces by I. A. Shvedov (unpublished), for an arbitrary ring with identity Λ and any sheaf of Λ-modules F: the cohomology groups

$$H^i(X; F) = 0 \quad \text{for } i > \dim_\Lambda X.$$

We indicate that Lemma 3 also follows from a result of A. V. Zarelua ([4], Lemma 2).

We shall say that a compactum Y satisfies the condition (m, l, p) if $\dim Y = m$ and there exists in the space Y a base $\{V_\alpha\}$ of open sets–

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* The group $H^*(\hat{X} \times \hat{Y})$ could have been computed by O' Neill' s formula for the cohomology of a product of paracompact spaces (see ([9])). However, this formula, like the second formula in ([9]) for the cohomology of the product of a

paracompact space by a compactum, is erroneous. Possibly the first formula remains valid for the product of a paracompact space by a compactum. However, we prefer not to prove O' Neill' s formula in this case, but to obtain the statement we need in another way. The cohomological dimension of a product of paracompact spaces (see the corollary 1) was studied by Kodama. But in his paper ([6]) the incorrect formulas of O' Neill are substantially used, and therefore the corresponding results of that paper cannot be regarded as proved.

it is known that for any of its elements V_α the group $H^i(Y, Y \setminus V_\alpha)$ is isomorphic to a direct sum of groups Z_p for $l < i \leq m$. The product

$$\underbrace{P \times P \times \cdots \times P}_{N \text{ times}}$$

of two-dimensional compacta of L. S. Pontryagin satisfies the condition $(2N, N, p)$.

Theorem. *For every prime p and arbitrary integer N there exists a compactum $Y = Y_N(p)$ such that, for every paracompact space X whose dimension satisfies $\dim X \leq N$, the equality*

$$\dim_{Z_p} X = \dim(X \times Y) - \dim Y$$

holds.

Proof. We shall establish that every compactum Y satisfying the condition (m, l, p) is a test space P_N for the cohomological dimension $\dim_{Z_p}$ of paracompact spaces when $m - l > N$.

Let Γ_β be a base of all open sets of the paracompact space X, and let $\{V_\alpha\}$ be the base referred to in the condition (m, l, p) for the space Y. The system of sets $\{\Gamma_\beta \times V_\alpha\}$ forms a large base of the space $X \times Y$. The dimensions $\dim X$ and $\dim_Z X$ coincide for a paracompact space X, if $\dim X < \infty$. This theorem for compacta was proved by P. S. Aleksandrov; for paracompacta it was proved by Dowker ((3), Theorems 3.6, 5.2). By Lemma 2,

$$\dim_Z(X \times Y) = \max_{\alpha,\beta}\{n : H^n(X \times Y, (X \times Y) \setminus (\Gamma_\beta \times V_\alpha)) \neq 0\}.$$

Let us now verify the fulfillment of the conditions of Lemma 1 for $A = X \setminus \Gamma_\beta$, $B = Y \setminus V_\alpha$, $s = \dim_{Z_p} X$, and $t = l$; we have $s + m \geq m > l + N \geq t + n$. The group $G_q = H^q(Y, B)$ for $q > t$ is isomorphic to a direct sum of groups Z_p. By Lemma 3, $\dim_{G_q} X = \dim_{Z_p} X = s$. Consequently, the first group of the conditions of Lemma 1 is fulfilled, and therefore

$$H^i(X \times Y, (X \times Y) \setminus (\Gamma_\beta \times V_\alpha)) = 0$$

for $i > s+m$, i.e.

$$\dim X = \dim_Z X \leq s+m = \dim_{Z_p} X + \dim Y.$$

Thus,

$$\dim_{Z_p} X \geq \dim X - \dim Y.$$

To prove the opposite inequality, find in the space X an open set Γ_β such that $H^s(X, X \setminus \Gamma_\beta; G_m) \neq 0$, where $s = \dim_{Z_p} X = \dim_{G_m} X$. Now, for $s > 0$, all the conditions of Lemma 1 are satisfied. Consequently,

$$H^{m+s}(X \times Y, (X \times Y) \setminus (\Gamma_\beta \times V_\alpha)) \neq 0$$

and

$$\dim(X \times Y) \geq m+s = \dim_{Z_p} X + \dim Y.$$

It remains to consider the case $s = 0$. By a theorem of Kodama ((6), Corollary 1), in this case $\dim X = 0$. But then $\dim(X \times Y) = \dim Y$, and the assertion of the theorem is obvious. The theorem is proved.

Corollary 1. *Let X be a paracompactum, K a compactum,* $\dim X < \infty$*,* $\dim K < \infty$*. Then*

$$\dim_{Z_p}(X \times K) = \dim_{Z_p} X + \dim_{Z_p} K$$

for every prime number p.

Proof. Let Y be a space possessing the property (m, l, p) with $m - l > \dim X + \dim K$. Then the space $Y \times K$ will possess the property (m', l', p), where $m' = \dim(Y \times K) \geq m$, and $l' = l + \dim K$. Since $m' - l' \geq m - l - \dim K \geq \dim X$, $Y \times K$ will be a test space for the cohomological dimension of the space X. Therefore

$$\dim_{Z_p}(X \times K) = \dim(X \times K \times Y) - \dim Y = \dim_{Z_p} X + \dim(K \times Y) - \dim Y = \dim_{Z_p} X + \dim_{Z_p} K.$$

Corollary 2 *(Hurewicz theorem for cohomological dimension). Let $f : Z \to X$ be a closed mapping of a paracompact space Z onto a paracompact space X. If* $\dim Z < \infty$ *and* $\dim X < \infty$*, then*

$$\dim_{Z_p} Z \leq \dim_{Z_p} X + \max\{\dim_{Z_p} f^{-1}(x) : x \in X\}.$$

Proof. Let $\dim X = n_1$, $\dim Z = n_2$, let the compactum Y_1 have property (m_1, l_1, p) with $m_1 - l_1 > n_1$, and let the compactum Y_2 have property (m_2, l_2, p) with $m_2 - l_2 > n_2$. The compactum $Y_1 \times Y_2$ will then have property $(m_1 + m_2, l_1 + l_2, p)$, and therefore will be a test space for the cohomological dimension of the space Z. Consider the evident mapping $g : Z \times Y_1 \times Y_2 \to X \times Y_1$. By the Hurewicz theorem for dimension (see E. G. Sklyarenko [10]) we have

$$\dim(Z \times Y_1 \times Y_2) \leq \dim(X \times Y_1) + \max\{\dim g^{-1}(x, y) : x \in X,\ y \in Y_1\}.$$

By Theorem 1 this inequality can be rewritten in the following form:

$$\dim_{Z_p} Z + \dim(Y_1 \times Y_2) \leq$$

$$\leq \dim_{Z_p} X + \dim Y_1 + \max\{\dim_{Z_p} f^{-1}(x) + \dim Y_2 : x \in X\},$$

i.e.

$$\dim_{Z_p} Z \leq \dim_{Z_p} X + \max\{\dim_{Z_p} f^{-1}(x) : x \in X\}.$$

By the same method one can easily obtain generalizations, to the case of cohomological dimension, of the Hurewicz theorem on finite-to-one mappings and of various sum theorems. However, these assertions are known for cohomological dimensions with respect to arbitrary coefficient groups (see [4,8]).

Test spaces can also be constructed for some other coefficient groups, for example for the field Q of rational numbers. It is not known whether test spaces exist for the cohomological dimensions of paracompact spaces with respect to an arbitrary coefficient group.

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Note: Figure translations are in progress. See original paper for figures.