INFINITE HERMITIAN MATRICES WITH OPERATOR COEFFICIENTS

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Abstract

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INFINITE HERMITIAN MATRICES WITH OPERATOR COEFFICIENTS

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This article is devoted to the study of the deficiency indices of symmetric operators generated in a certain Hilbert space $\mathfrak{H}$ by infinite Hermitian matrices with operator coefficients.

1. Let $\{H_k\}_{k=0}^{\infty}$ be a sequence of Hilbert spaces with scalar products $(\cdot,\cdot)_k$ and norms $\|\cdot\|_k$. The Hilbert space $\mathfrak{H}$ is defined as the direct sum

$$\mathfrak{H} = \sum_{k=0}^{\infty} \oplus H_k.$$

The elements of this space are written in the form of sequences $U = \{u_0, u_1, \dots, u_k, \dots\}$, where $u_k \in H_k$. The scalar product in $\mathfrak{H}$ is defined by the formula

$$[U, V] = \sum_{k=0}^{\infty} (u_k, v_k)_k.$$

Linear operators in $\mathfrak{H}$ are specified by infinite matrices

$$C = (C_{jk})_{j,k=0}^{\infty}$$

with operator coefficients, where C_{jk} is a linear operator with domain of definition $D(C_{jk})$, dense in the space H_k, and range in H_j. Moreover, for all j and k, $D(C_{jk}) \supset D(C_{kk})$. The matrices are assumed to be **Hermitian** and **finite**. The Hermitian property of C means that $C_{kj} = C_{jk}^*$ (where C_{jk}^* is the operator adjoint to C_{jk}). Finiteness means that for each j there exists an integer $m_j \geq 0$ such that $C_{jk} = 0$ for all k satisfying $|j-k| > m_j$. Matrices for which $m_j \equiv m < \infty$ and for all j there exist bounded operators $C_{j+m,j}^{-1}$ will be called operator C_m-matrices.

Defined on the set of finite vectors* with components $u_k \in D(C_{kk})$, the correspondence $U \to CU$ generates a symmetric operator in $\mathfrak{H}$. Its closure in $\mathfrak{H}$ will be denoted by C. The operator C, obviously, is symmetric with a domain of definition dense in $\mathfrak{H}$.

Below we study the deficiency indices of the operator C depending on the properties of the operators C_{jk}. The formulated theorems find application in the consideration of operators in the space of secondary quantization.

2. Known sufficient conditions for self-adjointness of the operator C ((1), p. 151; (2)) are a generalization of T. Carleman' s condition, formulated for ordinary Jacobi matrices (C_1-matrices with numerical coefficients). The theorem given below is a supplement to the results of (1,2) in the case when the operators $C_{k,k+j}$ ($j \neq 0$) are unbounded for every $j \neq 0$, or are bounded and for at least one $j \neq 0$ satisfy the condition

$$\sum_{k=0}^{\infty} \|C_{k,k+j}\|^{-1} < \infty.$$

* A vector is called **finite** if only a finite number of its components are different from zero.

Let I_k be the identity operator in H_k, and δ_{jk} the Kronecker symbol.

Theorem 1. *The operator C is self-adjoint if there exists a complex number σ such that the quantities*

$$\mu_{jk} = \|C_{jk}(C_{kk} - \sigma I_k)^{-1}\|$$

satisfy the estimates

$$\sup_k \sum_{j=0}^{\infty} (1 - \delta_{jk})\mu_{jk} < 1, \qquad \sup_j \sum_{k=0}^{\infty} (1 - \delta_{jk})\mu_{jk} < 1$$

and, uniformly in k,

$$\sum_{j=k+N}^{\infty} \mu_{jk} \to 0 \quad \text{as } N \to \infty.$$

The proof of Theorem 1 is based on the well-known lemma of T. Kato (4). It asserts that, under a perturbation of a self-adjoint operator T_0 by a symmetric operator V, the property of self-adjointness is preserved if the domains of definition of the operators T_0 and V are related by the inclusion $D(T_0) \subset D(V)$, and there exist constants a and b ($0 \leq a < 1,\ b > 0$) such that for every $f \in D(T_0)$ the inequality

$$\|Vf\| \leq a\|T_0 f\| + b\|f\|$$

holds.

An illustration of Theorem 1 may be the case when $H_k = L_2(-\infty, \infty)$ for all $k \geq 0$ and the operator C is generated by an operator C_m-matrix: $C_{kk} = -d^2/dx^2$, while for $k < j \leq k+m$

$$C_{kj} + \varphi_{kj}(x)\frac{d}{dx} + \psi_{kj}(x)$$

and $C_{jk} = C^*_{kj}$. All the indicated operators are defined in $L_2(-\infty, \infty)$ on closures of finite functions, and the complex-valued functions $\varphi_{kj}(x)$, $\varphi'_{kj}(x)$, and $\psi_{kj}(x)$ belong to $L_2(-\infty, \infty)$, and their norms are uniformly bounded with respect to all k and j.

3. The presence of nonzero deficiency indices of the operator C is established for the case when all spaces H_k coincide and all operators C_{jk} are bounded. Put $\dim H_k = d \leq \infty$. The following theorem generalizes the corresponding results of papers ([5],[6]).

Theorem 2. *Let the elements of the C_m-matrix be such that:*

$$1) \quad \sum_{k=0}^{\infty} \|C^{-1}_{k,k+m}\| < \infty;$$

2) for sufficiently large k $\qquad \|C^{-1}_{k,k+m}C_{k,k-m}\| \leq 1 - \nu/k, \qquad$ where $\nu > m$;

$$3) \quad \sum_{k=0}^{\infty} \|C^{-1}_{k,k+m}C_{k,k+j}\| < \infty \quad \text{for } |j| < m.$$

Then the operator C has deficiency indices (dm, dm).

4. Let now C belong to the class of C_1-matrices with matrix coefficients ($m \times m$-matrices), i.e. have the form

$$C_0 = \begin{pmatrix} A_0 & B_2 & & \\ B_0^* & A_1 & B_1 & \\ & & \ddots & \ddots \\ & & B_1^* & \ddots \\ & & & \ddots \end{pmatrix}, \tag{1}$$

where the unspecified elements are zero, and $\det B_k \neq 0$ $(k = 0, 1, \ldots)$. For such matrices, called by M. G. Krein ([7]) J_m-matrices, it is possible to supplement the results formulated in Theorems 1 and 2.

Let us introduce into consideration the equation with respect to $w(k)$

$$\det\left[B^*_{k-1} + (A_k - \lambda I)w(k) + B_k w^2(k)\right] = 0, \tag{2}$$

where λ is a complex number, and I is the identity matrix of order m. Number the roots of equation (2) in the order of increase of their moduli:

$$|w_1(k)| \leq |w_2(k)| \leq \cdots \leq |w_{2m}(k)|.$$

Denote by n_+ (n_-) the number of solutions $w(k)$ of equation (2) for $\lambda = i$ $(\lambda = -i)$ satisfying, for all sufficiently large k, the estimate $|w(k)| \le 1 - \nu/k$, and by p_+ (p_-) −

the number of solutions with the estimate $|w(k)| \ge 1 - 1/2k$. In view of the condition $\det B_k \ne 0$, equation (2) is equivalent to the equation

$$\det\left[w^2(k)I + w(k)F_1(k;\lambda) + F_2(k)\right] = 0,$$

where $F_1(k;\lambda) = B_k^{-1}(A_k - \overline{\lambda}I)$, $F_2(k) = B_k^{-1}B_{k-1}^*$.

Theorem 3. Suppose: 1) the elements of the matrices $F_1(k+1;\pm i) - F_1(k;\pm i)$ and $F_2(k+1) - F_2(k)$ are absolutely summable on the interval $(0 \le k \le \infty)$; 2) the limiting values of all $2m$ roots $w_j = \lim_{k\to\infty} w_j(k)$ are nonzero and distinct; 3) there exists $\nu > 1/2$ such that $n_\pm^\nu + p_\pm = 2m$. Then the defect indices of the operator C are equal to $(m - p_+, m - p_-)$.

The proof of Theorem 3 is based on computing asymptotic estimates, as $k \to \infty$, for the components of the vector $u_k = (x_{km}, x_{km+1}, \dots, x_{km+m-1})$, which is a solution of the equation

$$B_{k-1}^* u_{k-1} + A_{ku}k + B_{ku_{k+1}} = \overline{\lambda} u_k \quad (k \ge 1). \tag{3}$$

The plan of the proof is as follows: replace the system of second-order difference equations (3) by a system of first-order difference equations $y(k+1) = G_{ky}(k)$ (where $y(k)$ is a $2m$-dimensional vector), and reduce the resulting system to quasidiagonal form

$$t(k+1) = (\Gamma_k + \Gamma_{kD}k)t(k), \tag{4}$$

where Γ_k is a diagonal matrix, and the elements $d_{ij}(k)$ of the matrix D_k satisfy the condition

$$\sum_{k=0}^{\infty} |d_{ij}(k)| < \infty.$$

After this, it remains to use I. M. Rapoport' s theorem (([8]), p. 62) on the asymptotics of the solutions of equation (4) and to compute, for large k, the asymptotics of the products $\prod_{s=0}^{k} w_j(s)$, where $j = 1, 2, \dots, 2m$. This makes it possible to determine the number of solutions of equation (3) belonging to l_2.

Let us now consider the operator C_0, defined as the restriction of the operator C by means of the additional condition $u_0 = 0$. The defect subspace T_λ^0 of the operator C_0 consists of the quadratically summable solutions of equation (3). The dimension of the defect subspace T_λ of the operator C is determined by the formula $\dim T_\lambda^0 = \dim T_\lambda + m$, obtained on the basis of a lemma of I. M. Glazman (([9]), p. 47).

A particular case of J_m-matrices is given by C_m-matrices with numerical coefficients (see Sec. 1). However, instead of system (3), for C_m-matrices it is more convenient to write a single difference equation of order $2m$

$$\sum_{j=k-m}^{k+m} c_{kj} x_j = \overline{\lambda} x_k.$$

The corresponding characteristic equation has the form

$$\sum_{s=0}^{2m} f_s(k) w^{2m-s}(k) = 0, \tag{5}$$

where $f_s(k) = c_{k,k+m-s}/c_{k,k+m}$ for $s \neq m$, and $f_m^{(k)} = (c_{kk} - \overline{\lambda})/c_{k,k+m}$. We number the roots of equation (5) in increasing order of their moduli and introduce the numbers $n_\pm^\nu$ and $p_\pm$ exactly as was indicated above.

Theorem 4. Suppose the elements of the C_m-matrix are such that:

1)
$$\sum_{k=0}^{\infty} |f_s(k+1) - f_s(k)| < \infty \quad \text{for } s = 1, 2, \dots, 2m;$$

2) the limiting values of all $2m$ roots $w_j = \lim_{k\to\infty} w_j(k)$ of equation (5) are nonzero and distinct;

3) there exists $\nu > 1/2$ such that $n_\pm^\nu + p_\pm = 2m$.

Then the deficiency indices of the operator $\widetilde{C}$ are $(m - p_+, m - p_-)$.

As examples, let us consider operators C generated by C_m-matrices with elements of the form $c_{k,k+j} = a_j k^{\alpha_j}$ $(j = 0, 1, \dots, m)$. If $\alpha_0 = \alpha_m > \alpha_i + 1$ $(i = 1, 2, \dots, m-1)$ and $2|a_m| > |a_0|$, then C has deficiency indices (m, m), although it does not satisfy the conditions of Theorem 2. As an example of an operator with deficiency indices (ρ, ρ), where $0 < \rho < m$, one may take the case: $m = 2$, $c_{kk} = (a + 2a/k)k^\alpha$, $c_{k,k+1} = bk^\alpha$, $c_{k,k+2} = k^\alpha$, $\alpha > 1$, $|a + 2| < 2|b|$, $a < b^2/4 + 2$. The corresponding operator C has deficiency indices $(1, 1)$.

For ordinary Jacobi matrices the following theorem holds.

Theorem 5. Let C be the operator generated by a Jacobi matrix with elements $a_k = ak^\alpha$ and $b_k = bk^\beta$ (a and α are real, $\beta > 1$, b is complex). Then: 1) C is self-adjoint if $\alpha > \beta$, or if $\alpha = \beta$ and $|a| \geq 2|b|$; 2) C has deficiency indices $(1, 1)$ if $\alpha < \beta$, or if $\alpha = \beta$ and $|a| < 2|b|$.

5. The results presented find application in clarifying the question of self-adjointness of certain operators in the space of second quantization. Let, as usual, $a^*(p)$ and $a(p)$ be the creation and annihilation operators satisfying the Bose or Fermi commutation relations. The spaces H_k are symmetrized or antisymmetrized tensor products of k copies of the space $L_2(E_3)$, where E_3 is three-dimensional Euclidean space. With the aid of Theorem 1, for example, one verifies the self-adjointness of the operator $T = T_0 + V + V^*$, where

$$T_0 = \int_{E_3} \omega(p)a^*(p)a(p)\,dp + \int_{E_3}\int_{E_3} \varphi(p_1,p_2)a^*(p_1)a^*(p_2)a(p_1)a(p_2)\,dp_1dp_2,$$

$$V = \int_{E_3}\int_{E_3}\int_{E_3}\int_{E_3} v(p_1,p_2,p_3 \mid q)a^*(p_1)a^*(p_2)a^*(p_3)a(q)\,d^3p\,dq,$$

$$\omega(p) \geq 0, \qquad \varphi(p_1,p_2) \geq \delta > 0, \qquad v \in L_2(E_{12}), \qquad \|v\|\delta^{-1} < 1/2.$$

When considering the model of a system with one degree of freedom, i.e., when the argument p assumes a single value, one can use Theorem 4. For example, the operator C generated by the expression

$$a^*a + \varepsilon(\overline{\gamma}_2 a^{*4} + \overline{\gamma}_1 a^{*3}a + \gamma_0 a^{*2}a^2 + \gamma_1 a^*a^3 + \gamma_2 a^4)$$

is self-adjoint for arbitrary real ε and γ_0, if the roots of the equation

$$\gamma_2 w^4 + \gamma_1 w^3 + \gamma_0 w^2 + \gamma_1 w + \overline{\gamma}_2 = 0$$

are distinct and different from zero, and the moduli of two of them are less than 1 and of two are greater than 1. This holds if $|\gamma_0| \geq 2|\gamma_1|+2|\gamma_2|$. The deficiency indices of such an operator will be $(2,2)$, if $\gamma_1 = 0$ and $|\gamma_0| < 2|\gamma_2|$.

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Note: Figure translations are in progress. See original paper for figures.