MUTUAL RESTRICTIONS ON CURVATURE AND AREA FOR SURFACES LYING IN A COMPACT PART OF SPACE

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Abstract

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MUTUAL RESTRICTIONS ON CURVATURE AND AREA FOR SURFACES LYING IN A COMPACT PART OF SPACE

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It is natural to expect that if a closed surface of fixed area is contained in a small ball, then it has many folds and protrusions; more precisely, the variation of its Gaussian curvature is large. The results given below reflect this intuitive fact to some extent.

Let us introduce the following notation: F is a surface of class C^2, C^2-isometrically immersed in E^3; S is the area of F; p is the length of the boundary of F; d is the diameter of a ball containing F. Put

$$M = \int_F |K| \, dS, \qquad \omega^+ = \int_F K^+ \, dS,$$

where K is the Gaussian curvature of F, and the integration is with respect to surface area. Absolute positive constants will be denoted by the letter C with various subscripts. Admissible (not best possible) values of these constants are established in the course of the proofs.

Theorem 1. *If F is a closed surface, then*

$$S \leq C_1 M d^2. \tag{1}$$

It is clear that the right-hand side in (1) is equal to $2C_1(\omega^+ - \pi\chi)d^2$, where χ is the Euler characteristic of F. Inequality (1) may be regarded as a two-dimensional analogue of the well-known estimate of the length of a curve in terms of its diameter and the variation of its rotation ([1]).

Theorem 2. *If the surface F is homeomorphic to a disk and $\omega^+ < \pi$, then*

$$S \leq (C_2 + C_3 M)^{3/2} (p\sqrt{pd} + pd). \tag{2}$$

Corollary. *If the surface F is a simply connected geodesic disk of radius r and $\omega^+ < \pi$, then*

$$r \leq (C_4 + C_5 M)^6 d.$$

Therefore a complete surface homeomorphic to the plane, for which $\omega^+ < \pi$, $M < \infty$, is unbounded in E^3.

Remark. The requirement $\omega^+ < \pi$ is apparently not essential. The following simple theorem speaks in favor of this conjecture.

Theorem 3. *Let F be a locally convex surface (not necessarily of class C^2). Then*

$$S \leq C_6 (2 + \omega^+ - \chi)^3 d(p + d).$$

The proofs of Theorems 1-3 are obtained by a limiting transition from analogous theorems for polyhedral surfaces. These latter theorems seem to us to be of independent interest. We give their statements.

By a polyhedral surface $F = (R, \varphi)$ we mean an isometric immersion φ of a two-dimensional manifold R with a polyhedral metric into E^3 (or into E^2), under which every triangle of some

* What is meant is homeomorphism to a disk of the abstractly given surface, and not of the corresponding set in E^3.

of the triangulation T of the manifold R pass into a flat rectilinear triangle.

Let us introduce the positive and negative external curvatures μ^+, μ^- of the polyhedral surface F. Let $X \in \operatorname{int} R$, and let U_X be a neighborhood of X in R consisting of the triangles of the triangulation T adjacent to X. Denote by $P_X(\nu)$ the plane passing through the point $\varphi(X)$ with normal ν, and let $k(X, \nu)$ be the number of components of the set $\varphi(U_X) \setminus P_X(\nu)$. Put

$$n^+(\nu) = \sum_{k(X,\nu)=1} k(X, \nu), \qquad n^-(\nu) = \sum_{k(X,\nu)>2} \left(\frac{1}{2} k(X, \nu) - 1 \right),$$

$$\mu^+ = \frac{1}{2} \int n^+(\nu)\, d\sigma_\nu, \qquad \mu^- = \frac{1}{2} \int n^-(\nu)\, d\sigma_\nu,$$

where the integration is over the unit sphere.

It can be shown that the difference $\mu^+ - \mu^-$ is equal to the intrinsic curvature of F. Put, for the polyhedral surface, $M = \mu^+ + \mu^-$. Then the following assertions hold.

Theorem 1′. If F is a closed polyhedral surface, then estimate (1) holds.

Theorem 2′. If $F(R, \varphi)$ is a polyhedral surface, where the manifold R is homeomorphic to a disk and $\mu^+ < \pi$, then estimate (2) holds.

Remark. Let R be a closed two-dimensional manifold with a polyhedral metric, and let $\{F_n\}$ be a sequence of polyhedral surfaces $F_n = (R, \varphi_n)$ isometric to one another, with $\operatorname{diam} \varphi_n(R) \to 0$ (such a sequence always exists ([2])). Then it follows from Theorem 1′ that $\mu^+(F_n) \to \infty$.

The proof of Theorem 1′ is first carried out for specialized polyhedral surfaces in E^2 and is extended to the general case by the method of projections ([3]). The polyhedral surfaces used are characterized by the fact that no interior point of them is adjacent to more than two faces. Here, by the boundary of the surface $F = (R, \varphi)$ in E^2 we mean the maximal open set $Q \subset R$ such that each point $X \in Q$ has a neighborhood U which φ maps isometrically onto the set $\varphi(U)$.

The following plays the main role in the proof.

Lemma 1. Let F be a polyhedral surface in E^2 having only one face. Denote by $\tau(X)$ the turn of ∂R at a point X which is not a singleton component of ∂R. Then the estimate holds

$$S \leq C_4 d^2 \left\{ \sum_{\tau(X) \in [-\pi, \pi/2]} [\pi - \tau(X)] + \sum_{\tau(X) \in [-\pi, \pi/2]} |\tau(X)| \right\}.$$

In the proof of Theorem 2′, along with Lemma 1, the following is used.

Lemma 2. Let a geodesic broken line with endpoints A, B be contained in a boundary-convex domain $G \subset R$ homeomorphic to a disk, with $A, B \in \partial G$. Denote by Λ' the plane broken line with endpoints A', B', whose links are respectively equal to the links of Λ, and whose turns at the vertices of Λ' are equal to the turns at the corresponding vertices of Λ (all turns are taken on one side of the broken line). Then, if $\operatorname{diam} \Lambda' \leq \operatorname{diam} \Lambda = \rho(A, B)$ and the absolute curvature of the domain G is small (it is enough that $\operatorname{var} \omega(G) \leq 0.001$), then $A'B' > 0.1\rho(A, B)$, where ρ is the distance in the metric of R.

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Note: Figure translations are in progress. See original paper for figures.