ON EXTENDING THE RANGE OF APPLICATION OF SPECIAL ALGORITHMS OF LINEAR PROGRAMMING

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 512.251.26+519.3:330.115

MATHEMATICS

M. A. YAKOVLEVA

ON EXTENDING THE RANGE OF APPLICATION OF SPECIAL ALGORITHMS OF LINEAR PROGRAMMING

(Presented by Academician L. V. Kantorovich on 3 VI 1967)

We shall consider a linear programming problem in the following formulation. Vectors A_s $(s = 1, 2, \dots, N)$ and a vector P from the Euclidean space R^n are given, as well as real numbers c_s $(s = 1, 2, \dots, N)$. It is required to determine such nonnegative values of the variables x_s $(s = 1, 2, \dots, N)$ that

$$\sum_{s=1}^{N} x_s A_s = P$$

and the quantity

$$\sum_{s=1}^{N} c_s x_s$$

attains its minimum. In addition to general methods for solving the stated problem, a number of special methods have been developed which assume that the family of vectors A_s satisfies certain conditions. Some of these methods are constructed according to the following scheme. In the space R^n a set V is singled out in such a way that it is possible to construct certain simplified algorithms for solving systems of the form

$$(B_k, y) = d_k, \qquad k = 1, 2, \dots, n, \tag{1}$$

$$\sum_{k=1}^{n} z_k B_k = D, \tag{2}$$

where $D \in R^n$, d_k $(k = 1, 2, \dots, n)$ are arbitrary real numbers, and B_k $(k = 1, 2, \dots, n)$ form a linearly independent family of vectors from V. If all the vectors A_s in the formulation of the linear programming problem belong to the set V, then, in solving this problem by the method of successive improvement of an admissible vector ([1],[2]), one has to solve only systems of the form (1) and (2), and the existence of simplified algorithms makes it possible to increase substantially the volume of problems amenable to solution. For example, in order to construct algorithms for solving the general two-component problem ([3],[4]), V is taken to be the set of vectors in R^n in which no more than two components are different from zero. A number of special methods for solving the transportation problem and the problem with a block structure of the constraint matrix ([5]) also fit into this scheme.

In the present article we propose one general device which makes it possible to extend the range of application of special methods to the case where only part of the vectors A_s belongs to V. In what follows we shall assume that $A_s \in V$ for $s = 1, 2, \dots, M$, while the vectors $A_{M+1}, A_{M+2}, \dots, A_N$ are arbitrary. It is assumed here that the number $N - M$ is comparatively small.

As is known, one step of the method of successive improvement of an admissible vector consists in the following. At the beginning of the step a set $S \subset \{1, 2, \dots, N\}$ is singled out such that the family $\{A_s\}_{s\in S}$ forms a basis in R^n. A vector $y \in R^n$ is determined which satisfies the system

$$(A_s, y) = c_s, \qquad s \in S. \tag{3}$$

Then one finds $\sigma \in \{1, 2, \dots, N\}$ for which $(A_\sigma, y) > c_\sigma$, and the coefficients of the expansion of the vector A_σ in the basis $\{A_s\}_{s\in S}$, i.e., one solves the system

$$\sum_{s\in S} g_s A_s = A_\sigma. \tag{4}$$

After this, according to certain rules whose application does not require laborious computations, one determines the number $\tau \in S$ to be replaced by the number σ. The set $\bar{S} = (S \setminus \{\tau\}) \cup \{\sigma\}$ serves as the initial set for the next step. If the set S contains $s > M$, then the existing simplified algorithms are not directly applicable to systems (3) and (4). The main idea of the device described below is that the solutions of systems (3) and (4) are obtained with the aid of solutions of certain systems of the form (1) and (2).

Let $S = I \cup K$, where $I \subset \{1, 2, \dots, M\}$, $K \subset \{M + 1, M + 2, \dots, N\}$. Suppose, moreover, that some set $J \subset \{1, 2, \dots, M\}$ has been selected with a number of elements equal to the number of elements of the set K, and such that the family $\{A_s\}_{s\in I\cup J}$ forms a basis in R^n. Since the family $\{A_s\}_{s\in S}$ forms a basis in R^n, for $\mu \in J$

$$A_\mu = \sum_{\nu\in S} r_{\mu\nu} A_\nu.$$

We shall assume that, at the beginning of the step being described, some of the coefficients are known, namely, the coefficients $r_{\mu\nu}$ are known for $\mu \in J$, $\nu \in K$. Under these assumptions the solution y of system (3) can be found in the following way. We find the vector $\tilde{y}$ from the system

$$(A_\mu, \tilde{y}) = \begin{cases} c_\mu & \text{for } \mu \in I, \\ 0 & \text{for } \mu \in J. \end{cases} \tag{5}$$

Then the desired vector y can be found from the system

$$(A_\mu, y) = \begin{cases} c_\mu & \text{for } \mu \in I, \\ \sum\limits_{\nu \in K} r_{\mu\nu} \left[c_\nu - (A_\nu, \tilde{y})\right] & \text{for } \mu \in J. \end{cases} \tag{6}$$

Systems (5) and (6) are already systems of the form (1) and can be solved by the simplified algorithm.

After the number σ has been found, the solution of system (4) can be obtained with the aid of the solution of two systems of the form (2). For this we first find the numbers $\tilde{g}_s$ from the system

$$\sum_{\mu \in I \cup J} \tilde{g}_\mu A_\mu = A_\sigma. \tag{7}$$

Then, for $\nu \in K$, the desired coefficients g_ν can be found by the formulas

$$g_\nu = \sum_{\mu \in J} r_{\mu\nu} \tilde{g}_\mu.$$

The remaining coefficients g_ν (for $\nu \in I$) are found from the system

$$\sum_{\nu \in I} g_\nu A_\nu + \sum_{\nu \in J} p_\nu A_\nu = A_\sigma - \sum_{\nu \in K} g_\nu A_\nu,$$

and in the solution of this system all p_ν will turn out to be equal to zero.

As has already been said, in order to pass to the next step, one finds the number τ in the set S and sets $\bar{S} = (S - \{\tau\}) \cup \{\sigma\}$. In this case one obtains

the following partition of the set $\bar{S}$ into the sets $\bar{I}$ and $\bar{K}$:

$$\begin{aligned} &\bar{I} = (I \setminus \{t\}) \cup \{\sigma\}, \quad \bar{K} = K \setminus \{t\}, \qquad \text{if } \sigma \leq M, \\ &\bar{I} = (I \setminus \{t\}), \qquad\quad \bar{K} = (K \setminus \{t\}) \cup \{\sigma\}, \qquad \text{if } \sigma > M. \end{aligned}$$

In order that the step described can be repeated, it is also necessary to specify a new set $\bar{J}$ and the coefficients $\bar{r}_{\mu\nu}$ $(\mu \in \bar{J},\ \nu \in \bar{K})$ in the expansions

$$A_\mu = \sum_{\nu \in \bar{S}} \bar{r}_{\mu\nu} A_\nu, \qquad \mu \in \bar{J}. \tag{8}$$

As for the set $\bar{J}$, it is required only that the family $\{A_s\}_{s \in \bar{I} \cup \bar{J}}$ form a basis in R^n. If $\sigma > M$ and $\tau \in \bar{K}$, then $I = \bar{I}$, and one may put $\bar{J} = J$. If $\sigma > M$ and $\tau \in I$, then as $\bar{J}$ one may take the set $J \cup \{\tau\}$. In the case $\sigma \leq M$ the number σ enters the set $\bar{I}$, and some element a of the old set $I \cup J$ must be removed. From the rule for forming the set $\bar{I}$ it is clear that the required element a must belong to the set R, where $R = I$ if $\tau \in K$, and $R = J \cup \{t\}$ if $\tau \in I$. In this case the requirement of linear independence of the family $\{A_s\}_{s \in \bar{I}\bar{J}}$ is equivalent to the coefficient $\bar{g}_\alpha$ in the expansion (7) being different from zero. As a one may take that number for which $|\bar{g}_\alpha| = \max_{\mu \in R}(|\bar{g}_\mu|)$.

If the rule for choosing the number τ is taken into account, then one can show that in this case $\bar{g}_\alpha \neq 0$. Thus, $\bar{J} = R$ if $\sigma > M$, and $\bar{J} = R \setminus \{\alpha\}$ if $\sigma \leq M$.

The coefficients $\bar{r}_{\mu\nu}$ in the expansions (8) can be found by the usual formulas for transforming coefficients when passing to a new basis, i.e.,

$$\bar{r}_{\mu\nu} = \begin{cases} r_{\mu\nu} - \dfrac{g_\nu}{g_\tau} r_{\mu\tau}, & \text{for } \nu \in \bar{K} \setminus \{\sigma\}, \\ \dfrac{r_{\mu\tau}}{g_\tau}, & \text{for } \nu = \sigma \ (\text{if } \sigma \in \bar{K}). \end{cases} \tag{9}$$

This determines the coefficients $\bar{r}_{\mu\nu}$ for $\nu \in \bar{K},\ \mu \in \bar{J} \setminus \{\tau\}$. If $\tau \in \bar{J}$, then one must also put

$$\bar{r}_{\tau\nu} = \begin{cases} -\dfrac{g_\nu}{g_\tau}, & \text{for } \nu \in \bar{K} \setminus \{\sigma\}, \\ \dfrac{1}{g_\tau}, & \text{for } \nu = \sigma. \end{cases}$$

The right-hand sides of formulas (9) contain the coefficients $r_{\mu\tau}$. If $\tau \in I$, then, before applying formulas (9), it is necessary to compute the coefficients $r_{\mu\tau}$. For this one should solve the system

$$(A_\mu, z) = \begin{cases} 0, & \text{for } s \in (I \setminus \{t\}) \cup J, \\ -1, & \text{for } s = \tau, \end{cases} \tag{10}$$

which is a system of the form (1), and then put

$$r_{\mu\tau} = \sum_{\nu \in K} r_{\mu\nu}(A_\nu, z).$$

In conclusion we note that the number of elements of the set K cannot exceed $N - M$, and therefore the number of coefficients $r_{\mu\nu}$ stored at each step is no more than $(N - M)^2$. Moreover, the number of systems of the form (1) or (2) solved at each step is no more than five (system (10) need be solved only when $\tau \in I$), i.e., it does not depend on the number of elements of the set K.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

Received
25 V 1967

REFERENCES

1. L. V. Kantorovich, *Economic Calculation of the Best Use of Resources*, Publishing House of the Academy of Sciences of the USSR, 1959.

2. D. B. Yudin, E. G. Golshtein, *Linear Programming*, Moscow, 1963.

3. M. K. Gavurin, G. P. Rubinshtein, S. S. Shur, *Siberian Mathematical Journal*, 3, No. 4, 481 (1962).

4. M. A. Yakovleva, collection *Optimal Planning*, vol. 2, Novosibirsk, 1964, p. 23.

5. R. A. Zvyagina, collection *Optimal Planning*, vol. 2, Novosibirsk, 1964, p. 50.

Note: Figure translations are in progress. See original paper for figures.