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**Abstract**

**Full Text**

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## INDUCTIVELY REFLEXIVE LOCALLY CONVEX SPACES

**I. A. Berezanskii**

*(Presented by Academician P. S. Novikov, January 19, 1968)*

Let  $(E, \tau)$  be a locally convex space and let  $u = \{U_\alpha\}_{\alpha \in A}$  be a base of neighborhoods of zero in the topology  $\tau$ . For every neighborhood of zero  $U_\alpha \in u$ , we norm the linear hull  $L(U_\alpha^0)$  of its polar  $U_\alpha^0$ , taking  $U_\alpha^0$  as the unit ball (norm  $n_{U_\alpha^0}$ ). The inductive topology  $i_{E, \tau}$  in  $E'$  is defined as the strongest among all locally convex topologies  $t$  for which all identity embeddings

$$i_\alpha : (L(U_\alpha^0), n_{U_\alpha^0}) \rightarrow (E', t)$$

are continuous. This topology does not depend on the choice of a base of  $\tau$ -neighborhoods of zero and may be characterized as the strongest locally convex topology in  $E'$  for which all equicontinuous sets are bounded. The second dual  $(E', i_{E, \tau})'$  is the space of all functions from  $E^*$  (the algebraic dual of  $E'$ ) that are bounded on every equicontinuous subset of  $E' = (E, \tau)'$ .

We shall henceforth assume that the original space  $E$  is canonically embedded in  $(E', i_{E, \tau})'$  ( $e \mapsto \langle e, \cdot \rangle$ ). The space  $(E, \tau)$  is called inductively semireflexive if, under this embedding,  $E = (E', i_{E, \tau})'$ . If, in addition, the topology  $\tau$  of the space  $E$  coincides with the inductive topology  $i_{E', i_{E, \tau}}$ , then such a space is called inductively reflexive.

In the present paper criteria for inductive semireflexivity (and reflexivity) are given. It turns out that the class of inductively semireflexive spaces contains all complete spaces of type  $(S)$  (in particular, all complete nuclear spaces), as well as all semireflexive spaces of type  $(F)$  and  $(DF)$ . For an arbitrary locally convex space  $(E, \tau)$  a special class of topologies ( $\tau\rho$ -topologies) is defined. This class includes the maximal  $S$ -topology and the maximal nuclear topology majorized by  $\tau$ . It turns out that, if  $(E, \tau)$  is inductively semireflexive, then  $E$  is complete in any  $\tau\rho$ -topology.

**1.1.** A set  $A$  in a linear space  $E$  is called totally bounded with respect to a set  $B$  (notation  $A \prec B$ ) if, for every  $\varepsilon > 0$ , it can be covered by a finite number of translates of the set  $\varepsilon B$ .

A locally convex space  $(E, \tau)$  is called a space of type  $(S)$  (a Schwartz space  $(^1)$ ), and its topology an  $S$ -topology, if for every neighborhood of zero  $U$  there exists a neighborhood of zero  $V \prec U$ .

**1.2. Proposition.** *If  $(E, \tau)$  is a space of type  $(S)$ , then the second dual  $(E', i_{E, \tau})'$  is canonically isomorphic to its completion  $\widetilde{E}$ .*

**Proof.** Let  $e \in (E', i_{E, \tau})'$  and let  $U$  be an arbitrary  $\tau$ -neighborhood of zero. Since  $\tau$  is an  $S$ -topology, there exists a  $\tau$ -neighborhood of zero  $V \prec U$ . Then  $U^0 \prec V^0$  <sup>(3)</sup>. Since  $e$  is bounded on the polars of neighborhoods of zero, the restriction of  $e$  to  $L(V^0)$  is continuous in the norm  $n_{V^0}$ . Since the unit ball  $V^0$  of the space  $(L(V^0), n_{V^0})$  is closed in the topology  $\sigma(E', E)$ ,  $U^0$ , being complete in the uniformity induced by the topology  $\sigma(E', E)$ , is also complete in the uniformity induced in  $L(V^0)$  by the norm  $n_{V^0}$  <sup>(4)</sup>, p.31). Being, moreover, totally bounded in  $(L(V^0), n_{V^0})$ , the set  $U^0$  is bicomact. Hence the topology generated by the norm  $n_{V^0}$  coincides on  $U^0$  with the weak topology  $\sigma(E', E)$ . By Grothendieck's theorem <sup>(2)</sup>,  $\widetilde{E}$  is canonically identifiable with the space of all func-

functionals from  $E'^*$  whose restriction to the polar  $U^0$  of any  $\tau$ -neighborhood of zero  $U$  is continuous in the weak topology  $\sigma(E', E)$ . Thus,  $e \in \widehat{E}$ .

Conversely, let  $e \in \widehat{E}$  (where  $\widehat{E}$  is canonically embedded in  $E'^*$ ). Then the functional  $e$  must be bounded on every equicontinuous set from  $(E, \tau)'$ , since every such set is  $\sigma(E', \widehat{E})$ -bounded. Hence  $e \in (E', i_{E, \tau})'$ , and the proposition is proved.

1.3. For every locally convex space  $(E, \tau)$  there exists the strongest  $S$ -topology in  $E$  majorized by  $\tau$ . This topology was first considered by D. A. Raikov in <sup>(3)</sup> and was denoted there by  $k_0$ .

**1.4. Proposition.** *Let  $(E, \tau)$  be a locally convex space. Then in the conjugate space  $E'$  the inductive topologies  $i_{E, \tau}$  and  $i_{E, k_0}$  coincide.*

**1.5. Theorem** (criterion for inductive semireflexivity). *A locally convex space  $(E, \tau)$  is inductively semireflexive if and only if  $E$  is complete in the topology  $k_0$ .*

1.6. Since equicontinuous sets in  $E'$  are strongly bounded, we have  $i_{E, \tau} \geq \beta_E$ . On the other hand,  $\beta_E$  majorizes every topology compatible with the duality  $\langle E', E \rangle$ . But if  $(E, \tau)$  is inductively semireflexive, then the topology  $i_{E, \tau}$  itself is compatible with the duality  $\langle E', E \rangle$ . Thus, in this case  $i_{E, \tau} = \beta_E$ . It follows, in particular, that every inductively semireflexive space is semireflexive. Below (3.6) examples of spaces will be given for which these properties are equivalent.

In Komura's paper <sup>(5)</sup> an example was constructed of a Montel space  $(E, \tau)$  that is not complete. This space is semireflexive, but not inductively semireflexive, since otherwise it would be complete in the topology  $k_0$ , and a fortiori in its original topology (for  $(E, \tau)' = (E, k_0)'$ ).

**1.7. Theorem.** *A locally convex space  $(E, \tau)$  is inductively reflexive if and only if it is inductively semireflexive and bornological.*

2.1. Let  $(E, \tau)$  be a locally convex space. We shall say that a  $\tau$ -rotor topology

( $\tau r$ -topology)  $t$  is given on  $E$  if

$$\sigma(E, E') < t \leq \tau$$

and for every  $\tau$ -neighborhood of zero  $U$  there exists a sequence of nonzero elements  $\{e'_k\}_{k=1,2,\dots} \subset U^0$  such that every sequence  $\{\tilde{e}'_k\}_{k=1,2,\dots} \subset U^0$  satisfying

$$n_{U^0}(\tilde{e}'_k) = n_{U^0}(e'_k) \quad (k = 1, 2, \dots)$$

is equicontinuous with respect to  $t$ .

The topology  $k_0$  is  $\tau$ -rotor. Indeed, for every neighborhood of zero  $U$  it suffices to choose in  $U^0$ , as  $\{e'_k\}_{k=1,2,\dots}$ , any sequence converging to 0 in the norm  $n_{U^0}$  ( $e'_k \neq 0$ ). One can show that the strongest barreled topology in  $E$  majorized by  $\tau$  is also  $\tau$ -rotor.

The following assertion is a strengthening of 1.4.

**2.2. Proposition.** *If  $(E, \tau)$  is a locally convex space, then whatever  $\tau r$ -topology  $t$  on  $E$  we choose, the corresponding topology  $i_{E,t}$  on  $E'$  coincides with  $i_{E,\tau}$ .*

**Proof.** Since  $\tau \geq t$ , we have  $i_{E,\tau} \leq i_{E,t}$ . Now let  $V \subset E'$  be some neighborhood of zero in the topology  $i_{E,t}$ , and let  $U$  be an arbitrary  $\tau$ -neighborhood of zero. Consider any sequence

$$\{e'_k\}_{k=1,2,\dots} \subset U^0 \quad (e'_k \neq 0).$$

If we assume that  $V$  does not absorb  $U^0$ , then for every  $k = 1, 2, \dots$  there is an element  $\tilde{e}'_k$  such that

$$n_{U^0}(\tilde{e}'_k) = n_{U^0}(e'_k)$$

and

$$\tilde{e}'_k \notin kV.$$

Hence the sequence  $\{\tilde{e}'_k\}_{k=1,2,\dots} \subset U^0$  is not absorbed by  $V$ . Thus, if the topology  $t$  is  $\tau$ -rotor, then we can find an equicontinuous (with respect to  $t$ ) sequence not absorbed by  $V$ . Thus the initial assumption ( $i_{E,\tau} < i_{E,t}$ ) leads us to a contradiction.

**2.3. Proposition.** Let  $(E, \tau)$  be a space of type  $(S)$ . Then the completion of the space  $E$  in any  $\tau$ -rotor topology coincides with its completion in  $\tau$ .

**2.4. Theorem.** *If a space  $(E, \tau)$  is inductively semireflexive, then  $E$  is complete in every  $\tau$ -rotund topology (and in every locally convex topology consistent with the duality  $\langle E, E' \rangle$  and majorizing some  $\tau\rho$ -topology).*

3. Let us dwell in more detail on the relation between the inductive topology  $i_{E,\tau}$  and the strong topology  $\beta_E$ .

3.1. We shall call a locally convex space  $(E, \tau)$  **strongly regular** if every  $\sigma(\cdot, E')$ -bounded set in the second dual space  $(E', i_{E,\tau})'$  is contained in the  $\sigma(\cdot, E')$ -closure of some bounded set from  $(E, \tau)$  (if the topology  $i_{E,\tau}$  is replaced

by  $\beta_E$ , then we obtain the definition of a regular space). A strongly regular space is, obviously, regular.

**3.2. Proposition.** *A space  $(E, \tau)$  is strongly regular if and only if in  $E'$  the topologies  $i_{E, \tau}$  and  $\beta_E$  coincide.*

We give the proof of sufficiency. Let  $i_{E, \tau} = \beta_E$ . As we have already established in 1.2 and 1.4,  $(E', i_{E, \tau})'$  coincides with the completion  $\widehat{E}_{k_0}$  of the space  $(E, k_0)$ , and  $i_{E, \tau} = i_{E, k_0}$ . Further,

$$(E, \tau)' = (E, k_0)' = (\widehat{E}_{k_0}, \widehat{k}_0)'$$

( $\widehat{k}_0$  is the topology induced on the completion). The inductive topology  $i_{E, k_0}$  in  $E'$ , as is easy to see, coincides with  $i_{\widehat{E}_{k_0}, \widehat{k}_0}$ . For the topologies in  $E'$  the relations

$$\beta_{(E', i_{E, \tau})'} = \beta_{\widehat{E}_{k_0}} \geq \beta_E, \quad \beta_E = i_{E, \tau} \quad \text{and} \quad i_{\widehat{E}_{k_0}, \widehat{k}_0} \geq \beta_{\widehat{E}_{k_0}}$$

hold. Therefore  $\beta_E = \beta_{(E', i_{E, \tau})'}$ . Now let  $B \subset (E', i_{E, \tau})'$  be an arbitrary  $\sigma(\cdot, E')$ -bounded set.  $B^{E'}$  is a neighborhood of zero in the topology  $\beta_{(E', i_{E, \tau})'}$ . By the preceding equality we can find a bounded set  $A$  from  $(E, \tau)$  such that

$$A^{E'} \subset B^{E'}.$$

Hence,

$$B \subset B^{E'(E', i_{E, \tau})'} \subset A^{E'(E', i_{E, \tau})'},$$

which was required to be proved.

**3.3. Proposition.** *In order that the space  $(E, \tau)$  be strongly regular, it is sufficient that the following conditions be satisfied.*

- 1) *The strong dual  $(E', \beta_E)$  is bornological.*
- 2) *Every bounded sequence  $\{e'_k\}_{k=1, 2, \dots} \subset (E', \beta_E)$  is equicontinuous.*

**3.4. Corollary.** *Every space of type (DF) is strongly regular. An (F)-space is strongly regular if and only if it is regular.*

**3.5. Theorem.** *A locally convex space  $(E, \tau)$  is inductively semireflexive if and only if it is semireflexive and strongly regular.*

**3.6. Corollary.** *For spaces of type (F) and (DF), inductive (semi)reflexivity is equivalent to (semi)reflexivity. Hence it follows in turn that a semireflexive space of type (F) or (DF) is complete in every  $\tau$ -rotund topology (and in every locally convex topology consistent with the duality  $\langle E, E' \rangle$  and majorizing some  $\tau\rho$ -topology).*

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*Note: Figure translations are in progress. See original paper for figures.*

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