ON SINGLE-MODE OPEN RESONATORS

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V. P. MASLOV, E. M. VOROB' EV

ON SINGLE-MODE OPEN RESONATORS

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As early as 1958, one of the authors [1] studied the asymptotics, as $k \to \infty$, of the solution of the plane problem

$$\Delta\psi + k^2\psi = 0 \tag{1}$$

with boundary conditions on the equidistant curves Γ_1, Γ_2, separated from one another by a distance a:

$$\psi\big|_{\Gamma_1} = \psi\big|_{\Gamma_2} = 0. \tag{2}$$

It was found that the asymptotics of a series of eigenfunctions and eigenvalues corresponding to the geometrical-optics motion of rays along the normals to the curves is expressed in the form

$$k = \sqrt{(\pi n/a)^2 + \gamma^2} + O([a\gamma/n]^3),$$

$$\psi = \frac{\sin(\pi n r/a)}{\sqrt{[1+\varkappa(s)r]\,[1+\varkappa(s)a]^{-1}}}\, f_\gamma(s) + O\Big(\frac{a\gamma}{n}\Big), \tag{3}$$

where n is an integer; s is the arc length of the curve Γ_1; $\varkappa(s)$ is the curvature of the curve Γ_1; r is the length of the normal to it; and $f_\gamma(s)$ and γ satisfy the equation

$$-f_\gamma'' + \left\{-\frac{\varkappa^2}{4} - \frac{\varkappa''}{4}\,\frac{a}{1+\varkappa a} + \frac{(\varkappa')^2}{3}\,\frac{a^2}{(1+\varkappa a)^2} - \gamma(1+\varkappa a)\right\} f_\gamma = 0. \tag{4}$$

In the case when the curves Γ_1 and Γ_2 extend to infinity, equation (4), under the condition $\int f_\gamma^2 ds < \infty$, gives the asymptotics as $k \to \infty$ of all quasistationary eigenvalues of problem (1)–(2); they correspond to the natural oscillations in an open plane resonator of electromagnetic waves with perfectly conducting walls Γ_1 and Γ_2.

Figure 1. Graph of the coefficient $b(\gamma, s)$ for f_γ in equation (4) (γ negative); $b_{\min} = -\frac{1}{4R^2} - \frac{\gamma(R+a)}{R}$.

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Fig. 2. Resonator I

Figure 2: Fig. 2. Resonator I

Fig. 1. Graph of the coefficient $b(\gamma, s)$ for f_γ in equation (4) (γ negative);

$$b_{\min} = -\frac{1}{4R^2} - \frac{\gamma(R+a)}{R}$$

Here the number n is the number of the longitudinal mode, and γ is the value of the transverse mode. If the curve Γ_1 consists of an arc of a circle of radius R, smoothly* passing into straight lines, then the coefficient of f_γ in (4) has the form of a one-dimensional potential well (Fig. 1).

By choosing l and R, we can always ensure that there exists only one eigenvalue of equation (4), i.e. that the resonator has only one transverse mode. To investigate this case it is more convenient to reduce (4) to the Schrödinger equation by making the substitution $d\tau = \sqrt{1 + \varkappa a}\, ds$, after which the equation for $u = (1 + \varkappa a)^{1/4} f$ will be

* Physically this means that the curvature changes little over distances of the order of the wavelength.

have the form

$$-\frac{d^2u}{d\tau^2} + \left[\frac{1}{48}\frac{(\varkappa')^2 a^2}{(1+\varkappa a)^3} - \frac{\varkappa^2}{4(1+\varkappa a)}\right] u = \gamma u. \tag{4*}$$

Of interest is the question of how the resonator responds to small changes (of the order of a longitudinal wavelength) in the curvature of the mirrors. In the case where equidistance is preserved, the system will respond only weakly even to substantial (in comparison with the wavelength) changes in the curvature of the mirrors. Otherwise, the change in the character of the spectrum will depend on the ratio between the magnitude of the curvature perturbation and the depth of the potential well (Fig. 1). For a sufficiently deep well, i.e., for sufficiently small R and a ($R, a \approx$ 1–5 mm), the system will respond only weakly to deviations from equidistance of the order of the wavelength.

To get rid of superfluous transverse modes in the case of a deep potential well (Fig. 1), one may consider segments of equidistant curves of such a length that the fundamental

Fig. 3. Resonator II

Figure 3: Fig. 3. Resonator II

Fig. 2. Resonator I

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transverse mode has a much higher Q than the other transverse modes. The corresponding ratios of the Q' s for the modes γ_1 and γ_2 are of the order of the ratio of the corresponding eigenfunctions $f_{\gamma_1}(s)$ and $f_{\gamma_2}(s)$ near the ends of the resonator.

An analogous situation occurs when the equidistant curves Γ_1 and Γ_2 are rotated about the z-axis, forming equidistant surfaces of revolution $\rho = \xi(s)$, $z = \eta(s)$. In this case, in cylindrical coordinates (z, ρ, φ), the three-dimensional equation (1) has the form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial z^2} - \frac{m^2}{\rho^2}\psi + k^2\psi = 0, \tag{1*}$$

and after the substitution $\psi = w/\sqrt{\rho}$ it reduces to

$$\psi_{\rho\rho} + \psi_{zz} + \frac{1/4 - m^2}{\rho^2}\psi + k^2\psi = 0.$$

Accordingly, the term

$$\frac{m^2 - 1/4}{\xi(\xi + a\eta')}\,u$$

is automatically added to equation (4*). It is not hard to see that, for a certain choice of parameters, the corrected equation (4*), corresponding to the surface of revolution shown in Fig. 2 (resonator I), will have one eigenvalue, and only for $m = 0$ (or for $m = 0$ and $m = 1$). In addition, because of the added term, for $m = 0$ there will be a point of the spectrum also for the surface of revolution shown in Fig. 3 (a coaxial cylinder*,

* On one side these resonators may be bounded by a mirror perpendicular to the generatrix. Then the corresponding equations (4) and (6) must be considered in a half-space.

smoothly joined to two coaxial cones–resonator II). With a proper choice of the parameters, in this case as well the supplemented equation (4*) will have only one point of the spectrum, and only for $m = 0$.

In this case the depth of the potential well corresponding to such a resonator can be increased merely by decreasing the inner diameter of the cylinder; consequently, in contrast to the preceding case, a deep potential well is possible even for a large distance a between the cylinders. All the arguments concerning perturbations of the curvature of the mirrors remain valid also for the case of surfaces of revolution.

The indicated surfaces possess the same properties when used as resonators of electromagnetic oscillations.

From Maxwell' s equations $\operatorname{rot}\mathbf{H} = ik\mathbf{E}$, $\operatorname{rot}\mathbf{E} = -ik\mathbf{H}$, $\operatorname{div}\mathbf{E} = 0$, it follows that $\operatorname{rot}\operatorname{rot}\mathbf{E} - \operatorname{grad}\operatorname{div}\mathbf{E} = k^2\mathbf{E}$.

In the coordinate system (s, r, φ), these equations for the E_s- and E_φ-components have the form

$$\Delta E_\varphi - \frac{E_\varphi}{(\xi + r\eta')^2} + 2\xi'\frac{1}{(\xi + r\eta')^2}\frac{\partial E_s}{\partial\varphi} + 2\frac{\eta'}{(\xi + r\eta')^2}\frac{\partial E_r}{\partial\varphi} = 0,$$

$$\Delta E_s + \left[-\frac{\varkappa^2}{(1+\varkappa r)^2} + \frac{(\eta')^2 - 1}{(\xi + r\eta')^2}\right]E_s - \frac{2\xi'}{(\xi + r\eta')^2}\frac{\partial E_\varphi}{\partial\varphi} + \frac{2\varkappa}{(1+\varkappa r)^2}\frac{\partial E_r}{\partial\rho} + \frac{E_r}{(1+\varkappa r)}\frac{\partial}{\partial s}\left[\frac{\eta'}{(\xi + r\eta')} + \frac{\varkappa}{1+\varkappa r}\right] = 0. \quad (5)$$

In deriving these equations we used the relations

$$\rho = \xi(s) + r\eta'(s); \qquad z = \eta(s) - r\xi'(s); \qquad \varkappa = -\frac{\xi''}{\eta'} = \frac{\eta''}{\xi'};$$

$$\frac{\partial\rho}{\partial s} = \xi'(1 + \varkappa r).$$

It is not difficult to see that the component E_r and its derivatives with respect to φ and s are of order $O(1/k)$, and therefore in system (5) one may neglect the terms containing $\partial E_r/\partial s$, $\partial E_r/\partial\varphi$, E_r. In the case $m = 0$, i.e., when $\partial E_s/\partial\varphi = \partial E_r/\partial\varphi = 0$, the system splits into two equations. The first equation coincides with the scalar equation (1*) for $m = 1$, and therefore everything said about resonator I in the scalar case is valid for it; the second contains (in comparison with the scalar case) the additional term

$$\left[-\frac{\varkappa^2}{(1+\varkappa r)^2} + \frac{(\eta')^2 - 1}{(\xi + r\eta')^2}\right]E_s.$$

Fig. 4. Potential well for the transverse mode of resonator II:

$$V(\tau) = \frac{3/4 - (\eta')^2}{\xi(\xi + \eta' a)} + \frac{3}{4}\frac{\varkappa^2}{1 + \varkappa a} + \frac{1}{48}\frac{(\varkappa')^2 a^2}{(1+\varkappa a)^3}.$$

It corresponds to an addition to equation (4*) of the form

$$\frac{\varkappa^2}{1+\varkappa a} - \frac{(\eta')^2-1}{\xi(\xi+a\eta')}.$$

This term vanishes in the cylindrical part of the resonator and increases the potential for the conical part of this resonator, so that the potential well corresponding to it is deeper than in the scalar case (see Fig. 4). This means that the parameters for which the problem has only one transverse mode will vary within different limits than in the scalar case.

It is not difficult to see that for $m > 1$ resonator I has no eigenvalues, and for $m = 1$ it has a mode of much lower Q than for $m = 0$. Resonator II has no modes for $m > 0$. This conclusion can be draw-

···solve, proceeding from a system of equations analogous to (4*) and corresponding to (5):

$$\left[-\frac{d^2}{d\tau^2} + \frac{m^2+\frac{3}{4}}{\xi(\xi+a\eta')} - \frac{\chi^2}{4(1+\chi a)} + \frac{1}{48}\frac{(\chi')^2 a^2}{(1+\chi a)^3} + \left\{\frac{\chi^2}{1+\chi a} - \frac{(\eta')^2}{\xi(\xi+a\eta')}\right\}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + \frac{2\xi' m}{\xi(\xi+a\eta')}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\mathbf{u} = \tag{6}$$

To select a single longitudinal mode in the indicated resonators, it is sufficient to place between the resonator mirrors a semitransparent mirror which is a surface of revolution of a curve equidistant with respect to the generators of the resonator, and which is separated from one of the mirrors by a distance

$$a_1 = \frac{n_1}{n}a,$$

where n_1 has no common divisors with n.

The single-mode resonators proposed here, unlike all those known previously, may, for a certain relation between the resonator parameters, have an arbitrarily high quality factor.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.