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## Abstract

## Full Text

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*MATHEMATICS*

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# ON THE QUASI-EQUIVALENCE OF BASES IN FINITE CENTERS OF HILBERT SCALES

*(Presented by Academician L. V. Kantorovich on 24 VII 1967)*

Let  $\{x_k\}$  and  $\{y_k\}$  be two bases in a linear topological space  $E$ . The basis  $\{y_k\}$  is said to be quasi-equivalent to the basis  $\{x_k\}$  if there exist a permutation  $\{n_k\}$  of the natural series, a system of numbers  $\sigma_n$ , and an isomorphism  $K$  of the space  $E$  onto itself such that

$$x_k = \sigma_{n_k} K(y_{n_k}). \quad (1)$$

It is known <sup>(1)</sup> that in a Hilbert space all unconditional bases are quasi-equivalent to one another (and in (1) one may dispense with the permutation). On the other hand, the quasi-equivalence of all bases\* has been established for the so-called centers of nuclear Hilbert scales <sup>(2)\*\*</sup> (in this case the permutation is already necessary). In the present paper the quasi-equivalence of unconditional bases in finite centers of completely continuous scales (not necessarily nuclear) is proved. Thus one of the problems posed in <sup>(2)</sup> is solved. The method proposed in the present paper not only makes it possible to dispense with the requirement that the space be nuclear, but also substantially simplifies the proof of quasi-equivalence (cf. <sup>(2)</sup>, <sup>(3)</sup>). The main theorem will be proved in Sec. 4°.

1°. Let  $E$  be a complete countably Hilbert space and let  $\|x\|_p$ ,  $p = 1, 2, \dots$ , be a system of Hilbert norms defining the topology  $\mathfrak{T}$  of the space  $E$ .

Under the indicated assumptions on the space  $E$ , the following holds.

**Lemma 1.** Let  $\{x_k\}$  be an unconditional basis in  $E$  and let  $\{x'_k\}$  be the biorthogonal system in  $E^*$ . Then the system of Hilbert norms

$$\|x\|_{0,p} = \left( \sum |x'_k(x)|^2 \|x_k\|_p^2 \right)^{1/2}$$

defines on  $E$  the original topology  $\mathfrak{T}$ .

This lemma is proved by applying Mazur's result <sup>(4)</sup> that unconditional convergence of the series  $\sum x_k$  in a Hilbert space implies  $\sum \|x_k\|^2 < \infty$ , and by the usual arguments connected with the closed graph theorem.

2°. Using the concept of an  $n$ -diameter and its simplest properties <sup>(5, 2)</sup>, it is easy to prove the following lemma.

**Lemma 2.** Let  $H_0 \subset H_1 \subset H_2 \subset H_3$  be a quadruple of Hilbert spaces with continuous embeddings;  $\{e_k\}$  a common orthonormal basis in  $H_0$  and  $H_3$ ;  $\{h_k\}$  a common orthonormal basis in  $H_1$  and  $H_2$ ;  $\|e_k\|_{H_0} = \|h_k\|_{H_1} = 1$ ,  $\|e_k\|_{H_3} = \mu_k \downarrow 0$ ,  $\|h_k\|_{H_2} = \nu_k \downarrow 0$ . Then there exists a constant  $C < \infty$  such that:

$$\mu_k \leq C\nu_k.$$

The assumption on the existence of bases  $\{e_k\}$  and  $\{h_k\}$  with the properties indicated in the lemma is equivalent to the assumptions that the embedding operator from  $H_1$  into  $H_2$  is completely continuous and that  $H_0$  is dense in  $H_3$ .

\* In a nuclear space all bases are absolute and, a fortiori, unconditional <sup>(2)</sup>.

\*\* The proof in <sup>(2)</sup> is carried out essentially by the method first applied in <sup>(3)</sup> in proving the quasi-equivalence of bases in the space  $A_r$ .

3°. Let  $E_\alpha = E_\alpha(a_k) = \lim_{\lambda < \alpha} \text{pr } H_\lambda$  be a finite (i.e.  $\alpha < \infty$ ) center of the Hilbert scale  $H_\lambda = l_2(a_k^\lambda)$ ,  $-\infty < \lambda < \infty$ ;  $a_k \uparrow \infty$ —this condition ensures the complete continuity of the embedding  $H_\lambda$  into  $H_{\lambda_1}$ ,  $\lambda_1 < \lambda$ , i.e. the space  $E_\alpha$  will be Montel <sup>(6)</sup>.\*

Let  $\{e_k\}$  be a system of unit vectors in  $H_0 = l_2$ ; then

$$\|x\|_\lambda = \|x\|_{H_\lambda} = \left( \sum |(x, e_k)|^2 a_k^{2\lambda} \right)^{1/2},$$

where  $(x, y) = (x, y)_{l_2}$ . Obviously,  $\{e_k\}$  is an unconditional basis in the space  $E_\alpha$ ; we shall call it the principal basis in  $E_\alpha$ . Finite centers possess the following important property, which will be used essentially below:

**Lemma 3.** Let the operator  $T \in L(E_\alpha, H_{\lambda_0})$ \*\* for some  $\lambda_0 < \alpha$ , and at the same time  $T \in L(H_\alpha, E_\alpha)$ . Then  $T \in L(E_\alpha, E_\alpha)$ .

This assertion is a consequence of the interpolation theorem for operators in Hilbert scales <sup>(7)</sup>.

4°. We shall prove the main theorem.

**Theorem.** Let the space  $E_\alpha$  satisfy the assumptions of item 3° and let  $\{x_k\}$  be an unconditional basis in  $E_\alpha$ . Then  $\{x_k\}$  is quasiequivalent to the principal basis  $\{e_k\}$  in  $E_\alpha$ .

**Proof.** For simplicity we put  $\alpha = 0$ . Let  $\{x'_k\} \subset E_0^*$  be the system biorthogonal to the basis  $\{x_k\}$ . By Lemma 1, the system of norms

$$\|x\|_{0,\lambda} = \left( \sum |x'_k(x)|^2 \|x_k\|_\lambda^2 \right)^{1/2}, \quad \lambda < 0,$$

is equivalent to the original system of norms  $\|x\|_\lambda$ ,  $\lambda < 0$ . Denote by  $H_\lambda^{(0)}$  the completion of the space  $E_0$  with respect to the norm  $\|x\|_{0,\lambda}$ . From the equivalence of the systems of norms it follows that there exists  $\lambda_0 < 0$  such that, for  $\lambda : \lambda_0 < \lambda < 0$ , numbers  $\mu_i = \mu_i(\lambda) < \infty$ ,  $i = 1, 2$ , can be found which ensure the following continuous embeddings:

$$H_{\mu_2} \subset H_\lambda^{(0)} \subset H_{\mu_1}. \quad (2)$$

Let now  $G_0$  be some Hilbert space for which:

- a) the inclusions  $H_0 \subset G_0 \subset E_0$  hold with continuous embeddings;
- b)  $\{x_k\}$  is an orthogonal basis in  $G_0$  (the existence of such a space  $G_0$  will be shown later).

Normalizing the system  $\{x_k\}$  in the space  $G_0$ , we obtain the system  $\{\sigma_k x_k\}$ ,  $\sigma_k > 0$ ,  $\|\sigma_k x_k\|_{G_0} = 1$ . Fix some  $\lambda : \lambda_0 < \lambda < 0$ , and renumber the system  $\{\sigma_k x_k\}$  so as to obtain a system  $y_k = \sigma_{n_k} x_{n_k}$ , ordered by decreasing norms in the space  $H_\lambda^{(0)}$ :  $\|y_k\|_{0,\lambda} \downarrow 0$ . It will be shown below that the linear operator  $T$ , defined by the relations

$$Tx = \sum_{k=1}^{\infty} (x, e_k)_0 e_k, \quad (3)$$

is an isomorphism of the space  $E_0$  onto itself. The latter will mean the quasiequivalence of the bases  $\{x_k\}$  and  $\{e_k\}$ , since  $e_k = \sigma_{n_k} K(x_{n_k})$ ,  $K = T^{-1}$ .

First we establish the existence of a Hilbert space  $G_0$  with the properties a) and b) indicated above. Let  $\mathfrak{M}$  be the set of all nondecreasing functions  $M(\lambda) > 0$ , defined on  $(-1, 0)$ . For each function  $M \in \mathfrak{M}$  consider the Hilbert space  $G(M)$  of all ele-

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\* The scale  $H_\lambda$  itself in this case is naturally called completely continuous.

\*\* The notation  $T \in L(X, Y)$  means that either the operator  $T$  maps  $X$  linearly and continuously into  $Y$ , or (if it is not defined on all of  $X$ ) it admits a closure to a linear continuous operator from  $X$  to  $Y$ .

ments from  $E$  having finite norm

$$\|x\|_{G(M)} = \left\{ \int_{-1}^0 \|x\|_{0,\lambda}^2 M(\lambda)^2 d\lambda \right\}^{1/2}.$$

It is not difficult to see that for every  $M \in \mathfrak{M}$  the space  $G(M)$  is continuously embedded in  $E_0$  and the system  $\{x_k\}$  is an orthogonal basis in  $G(M)$ . We shall show that one can choose  $M = M_0 \in \mathfrak{M}$  so that  $H_0$  is continuously embedded in  $G(M_0)$ . Indeed, the unit ball  $S$  of the space  $H_0$  is a bounded set in  $E_0$ ; therefore, for every  $\lambda < 0$  there is  $C(\lambda) < \infty$  such that

$$\|x\|_{0,\lambda} \leq C(\lambda) < \infty, \quad x \in S$$

(one may assume that  $C(\lambda)$  is increasing). Then  $S \subset U(M_0)$ , where  $M_0(\lambda) = C(\lambda)^{-1} \in \mathfrak{M}$ . The latter means that  $H_0$  is continuously embedded in  $G(M_0)$ . The space  $G_0 = G(M_0)$  thus constructed is the desired one.

Finally, let us show that the operator  $T$  (3) is an isomorphism of  $E_0$  onto itself. Since  $\{e_k\}$  and  $\{y_k\}$  are orthonormal bases, respectively, in  $H_0$  and  $G_0$ , the operator  $T$  is an isomorphism of the space  $H_0$  onto  $G_0$ . A fortiori,

$$T \in L(H_0, E_0), \quad T^{-1} \in L(H_0, E_0). \quad (4)$$

If we now establish that

$$T \in L(E_0, H_{\mu_1}), \quad T^{-1} \in L(E_0, H_{\mu_1}) \quad (5)$$

(for the definition of  $\mu_1$ , see (2)), then from (4) and (5), by Lemma 3, we obtain that  $T$  is an isomorphism of the space  $E_0$  onto itself.

Thus it remains to prove the relations (5). Applying Lemma 2 to the quadruple of spaces  $H_0 \subset G_0 \subset H_\lambda^{(0)} \subset H_{\mu_1}$  and to the bases  $\{e_k\}$  and  $\{y_k\}$ , we obtain

$$\|e_k\|_{\mu_1} \leq C \|y_k\|_{0,\lambda}. \quad (6)$$

From the same lemma, applied to the quadruple of spaces  $G_0 \subset H_{-\varepsilon} \subset H_{\mu_2} \subset H_\lambda^{(0)}$ ,  $\varepsilon > 0$ ,  $\mu_2 + \varepsilon < 0$ , and to the bases  $\{y_k\}$  and  $\{\gamma_k e_k\}$ ,  $\gamma_k = a_k^\varepsilon$ , it follows that

$$\|y_k\|_{0,\lambda} \leq C_1 \|\gamma_k e_k\|_{\mu_2} = C_1 \|e_k\|_{\mu_2+\varepsilon}, \quad C_1 = C_1(\varepsilon) < \infty. \quad (7)$$

From (6) and (7) we obtain the inequalities

$$\begin{aligned} \|Tx\|_{0,\lambda} &= \left( \sum |(x, e_k)|^2 \|y_k\|_{0,\lambda}^2 \right)^{1/2} \leq C_1 \left( \sum |(x, e_k)|^2 \|e_k\|_{\mu_2+\varepsilon}^2 \right)^{1/2} = \\ &= C_1 \left( \sum |(x, e_k)|^2 a_k^{2(\mu_2+\varepsilon)} \right)^{1/2} = C_1 \|x\|_{\mu_2+\varepsilon}, \end{aligned}$$

$$\begin{aligned} \|T^{-1}y\|_{\mu_1} &= \left( \sum |(x, e_k)|^2 a_k^{2\mu_1} \right)^{1/2} = \left( \sum |(x, e_k)|^2 \|e_k\|_{\mu_1}^2 \right)^{1/2} \leq \\ &\leq C \left( \sum |(x, e_k)|^2 \|y_k\|_{0,\lambda}^2 \right)^{1/2} = C \|y\|_{0,\lambda}, \quad x = T^{-1}y. \end{aligned}$$

These inequalities mean, in view of (2), that  $T \in L(H_{\mu_2+\varepsilon}, H_\lambda^{(0)}) \subset L(E_0, H_{\mu_1})$ ,  $T^{-1} \in L(H_\lambda^{(0)}, H_{\mu_1}) \subset L(E_0, H_{\mu_1})$ . The theorem is proved.

5°. Theorem 1 is equivalent to the following criterion for the isomorphism of the Köthe space  $L(b_{n,p})$  <sup>(2)</sup> to the finite center of a completely continuous Hilbert scale  $E_\alpha = E_\alpha(a_k)$ .

**Theorem 2.** In order that the space  $L(b_{n,p})$  be isomorphic to the space  $E_\alpha$ , it is necessary and sufficient that there exist a permutation  $\{n_k\}$  of the natural numbers and a sequence of numbers  $\sigma_k > 0$  such that the spaces  $E_\alpha$  and  $L(C_{k,p})$ ,  $C_{k,p} = \sigma_k a_{n_k,p}$ , coincide set-theoretically.

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*Note: Figure translations are in progress. See original paper for figures.*

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