ON A GENERAL BOUNDARY-VALUE PROBLEM

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Abstract

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FOR ONE SYSTEM OF EQUATIONS OF COMPOSITE TYPE

(Presented by Academician I. N. Vekua on 5 X 1967)

Let G be a simply connected domain in the plane $\zeta = \xi + i\eta$, bounded by a simple closed smooth curve γ, having no segments parallel to the axis $\xi = 0$, and possessing the property that every straight line $\xi = \text{const}$ passing through G intersects γ in exactly two points. Let N, M be the points of tangency of the curve γ with the straight lines $\xi = a$, $\xi = b$ $(a < b)$. Taking as positive on γ the direction of motion that leaves the domain G on the left, denote by γ_1 the arc $\widehat{MN}$. Consider in the domain G the system of equations

$$\begin{gathered}\partial\varphi/\partial\bar{\zeta} - \partial\varphi/\partial\zeta = A_0(\zeta)\varphi + B_0(\zeta)\bar{\varphi} + C_0(\zeta)\psi + D_0(\zeta)\bar{\psi},\\ \partial\psi/\partial\bar{\zeta} - q(\zeta)\partial\psi/\partial\zeta = A_1(\zeta)\varphi + B_1(\zeta)\bar{\varphi} + C_1(\zeta)\psi + D_1(\zeta)\bar{\psi}; \qquad \text{(I)}\\ \partial/\partial\bar{\zeta} = \tfrac{1}{2}(\partial/\partial\xi + i\partial/\partial\eta), \qquad \partial/\partial\zeta = \tfrac{1}{2}(\partial/\partial\xi - i\partial/\partial\eta)\end{gathered}$$

with respect to the complex-valued unknown functions $\varphi(\zeta)$, $\psi(\zeta)$, with given complex-valued coefficients. We shall assume that the coefficient $q(\zeta)$ of system (I) satisfies the condition $|q(\zeta)| \le q_0 < 1$, is Hölder continuous together with its derivatives up to order n' inclusive in the domain $\bar{G} = G + \gamma$, is Hölder continuous together with its derivatives up to order $n' - 1$ and belongs to the class L_p, $p > 2$, in the whole ζ-plane, while the coefficients $A_j(\zeta)$, $B_j(\zeta)$, $C_j(\zeta)$, $D_j(\zeta)$ will be regarded as Hölder continuous in $\bar{G}$ together with their derivatives up to order $n' - 1$, where n' is a certain positive integer, $n' = \max(m, n)$. System (I) is a system of composite type in the domain G with a double system of characteristics $\xi = \text{const}$. To a system of the form (I), in a neighborhood of each point of G, any system of the general form

$$\partial w_k/\partial\bar{\zeta} - q_k(\zeta)\partial w_k/\partial\zeta = \sum_{l=1}^{2}\left(A_{kl}(\zeta)w_l(\zeta) + B_{kl}(\zeta)\overline{w_l(\zeta)}\right), \qquad k = 1, 2,$$

with coefficients $q_k(\zeta)$ satisfying, in the indicated neighborhood, the conditions $|q_1(\zeta)| = 1$, $|q_2(\zeta)| \leq \text{const} < 1$, can be reduced.

Let m, n be positive integers. We pose the following problem:

Problem $D_{m,n}$. Find regular solutions of system (I) such that $\varphi(\zeta)$ is Hölder continuous up to order m inclusive, and $\psi(\zeta)$ is Hölder continuous up to order n inclusive in the closed domain $\bar{G}$, and satisfying the boundary conditions:

$$\operatorname{Re}\left(\sum_{j=0}^{m} a_j^{(k)}(\zeta)\frac{\partial^j \varphi}{\partial \xi^j} + \sum_{j=0}^{n} b_j^{(k)}(\zeta)\frac{\partial^j \psi}{\partial \zeta^j}\right) = h_k(\zeta), \qquad (D_{m,n})$$

where $\zeta \in \gamma$ for $k = 0$ and $\zeta \in \gamma_1$ for $k = 1, 2$, and $a_j^{(k)}(\zeta)$, $b_j^{(k)}(\zeta)$ are given complex-valued real functions respectively on γ for $k = 0$ and on γ_1 for $k = 1, 2$, Hölder continuous there. If pri-

take into account the required smoothness of the solutions of system (I) in the closed domain, then it is easy to see that the general boundary-value problem is reduced to the problem formulated:

$$\operatorname{Re}\left(\sum_{0\leq i+j\leq m} a_{ij}^{(k)}(\xi)\frac{\partial^{i+j}\varphi}{\partial \xi^i \partial \eta^j} + \sum_{0\leq i+j\leq n} b_{ij}^{(k)}(\xi)\frac{\partial^{i+j}\psi}{\partial \xi^i \partial \eta^j}\right) = h_k(\xi).$$

Let $\overline{\gamma}_1$ be the closure of the arc γ_1. The main results concerning the problem $D_{m,n}$ are contained in the following theorem:

Theorem. *If on the arc $\overline{\gamma}_1$ the inequality*

$$\Delta(\xi) = 2i\left(b_n^{(0)}(\xi)a_{12}(\xi) - b_n^{(1)}(\xi)a_{02}(\xi) + b_n^{(2)}(\xi)a_{01}(\xi)\right) \neq 0, \qquad \text{(N)}$$

$$a_{kl}(\xi) = \operatorname{Im}\left(a_m^{(k)}(\xi)a_m^{(l)}(\xi)\right),$$

is satisfied, on the arc $\gamma - \gamma_1$ the inequality $b_n^{(0)}(\xi) \neq 0$ is satisfied, and at the points M, N the equalities $a_j^{(0)} = 0$, $2a_{12} = -1$, $j = 0, 1, \ldots, m$, are satisfied, then the homogeneous problem $D_{m,n}^0$ ($h_k(\xi) \equiv 0$) can have only a finite number of linearly independent solutions, and for the solvability of the nonhomogeneous problem it is necessary and sufficient that a finite number of conditions be fulfilled; moreover, for the index of the problem $D_{m,n}$ the formula holds

$$\operatorname{Ind}(D_{m,n}) = 2(\varkappa + m + n) + 1,$$

where

$$\varkappa = \frac{1}{2\pi}\{\arg\Delta^*(\xi)\}_\gamma, \qquad \Delta^*(\xi) = \begin{cases} i\Delta(\xi), & \xi \in \overline{\gamma}_1, \\ b_n^0(\xi), & \xi \in \gamma - \gamma_1, \end{cases}$$

in which $\{\arg\Delta^*(\xi)\}_\gamma$ *denotes the increment of the function* $\arg\Delta^*(\xi)$ *when the contour* γ *is traversed once in the positive direction.*

In the general case the proof of this theorem is cumbersome. Therefore we shall restrict ourselves to the case when system (I) is split with respect to the functions $\varphi(\xi)$, $\psi(\xi)$, i.e., when $C_0(\xi) = D_0(\xi) = A_1(\xi) = B_1(\xi) \equiv 0$. In this case $\varphi(\xi) = e^{-ip(\xi)}\varphi^0(\xi)$, where the function $\varphi^0(\xi)$ is a solution of the Volterra integral equation

$$\varphi^0(\xi) - \int_{\eta_0}^{\eta} K_0(\xi + i\sigma)\overline{\varphi^0(\xi + i\sigma)}\, d\sigma = \omega(\xi),$$

$$K_0(\xi) = -iB_0(\xi)e^{2i\operatorname{Re}p(\xi)}, \qquad p(\xi) = \int_{\eta_0}^{\eta} A_0(\xi + i\sigma)\, d\sigma, \tag{1°}$$

$\omega(\xi)$ is an arbitrary complex-valued function, Hölder-continuous together with its derivatives up to order m inclusive on the segment $[a, b]$, and η_0 is fixed so that the line $\eta = \eta_0$ either touches the domain G or intersects it. By the method of successive approximations, from (1°) we find

$$\varphi^0(\xi) = \Gamma_1(\xi)\omega(\xi) + \Gamma_2(\xi)\overline{\omega(\xi)}, \tag{2}$$

where the functions $\Gamma_j(\xi)$ satisfy the condition

$$|\Gamma_1(\xi)|^2 - |\Gamma_2(\xi)|^2 = 1. \tag{3}$$

Hence

$$\varphi(\xi) = \Gamma_1^0(\xi)\omega(\xi) + \Gamma_2^0(\xi)\overline{\omega(\xi)}, \qquad \Gamma_j^0(\xi) = e^{-ip(\xi)}\Gamma_j(\xi). \tag{4}$$

Inequality (N) makes it possible to write the boundary conditions $(D_{m,n})$ in the form

$$\operatorname{Re}\left(i\Delta(\xi)\frac{\partial^n\psi}{\partial\xi^n} + \sum_{j=0}^{n-1}\Delta_j(\xi)\frac{\partial^j\psi}{\partial\xi^j} + \sum_{j=0}^{m-1}d_j(\xi)\frac{\partial^j\varphi}{\partial\xi^j}\right) = h^0(\xi), \qquad \xi \in \gamma_1; \tag{5}$$

$$\frac{\partial^m \varphi}{\partial \xi^m}+\sum_{j=0}^{m-1}\left(a_j(\xi)\frac{\partial^j \varphi}{\partial \xi^j}+b_j(\xi)\frac{\partial^j \overline{\varphi}}{\partial \xi^j}\right)=\sum_{j=0}^{n-1}C_j(\xi)\frac{\partial^j \psi}{\partial \xi^j}+\sum_{j=0}^{n}e_j(\xi)\overline{\left(\frac{\partial^j \psi}{\partial \xi^j}\right)}+h^{(1)}(\xi), \qquad \xi\in\gamma_1; \tag{6}$$

$$\operatorname{Re}\left(\sum_{j=0}^{m}a_j^{(0)}(\xi)\frac{\partial^j \varphi}{\partial \xi^j}+\sum_{j=0}^{n}b_j^{(0)}(\xi)\frac{\partial^j \psi}{\partial \xi^j}\right)=h_0(\xi), \qquad \xi\in\gamma-\gamma_1=\gamma_2, \tag{7}$$

where the functions $\Delta_j(\xi), d_j(\xi), a_j(\xi), b_j(\xi), c_j(\xi), e_j(\xi), h^0(\xi), h^{(1)}(\xi)$ are expressed in terms of the coefficients of condition $(D_{m,n})$ and can easily be written out.

Let $\eta=\sigma_j(\xi)$ be the equation of the curve γ_j in Cartesian coordinates. Then, taking into account equalities (3) and (4), from (6) for the function $\omega(\xi)$ we obtain an ordinary differential equation of order m:

$$\frac{d^m \omega}{d\xi^m}+\sum_{k=0}^{m-1}\left(A_k^0(\xi)\frac{d^k\omega}{d\xi^k}+B_k^0(\xi)\frac{d^k\overline{\omega}}{d\xi^k}\right)=f(\xi) \qquad (a\leqslant\xi\leqslant b), \tag{8}$$

with right-hand side

$$f(\xi)=\sum_{k=0}^{n-1}C_k^0(\xi)\left(\frac{\partial^k\psi}{\partial\xi^k}\right)_{\zeta=\xi+i\sigma_1(\xi)}+\overline{D_k^0(\xi)}\left(\frac{\overline{\partial^k\psi}}{\partial\overline{\xi}^k}\right)_{\zeta=\xi+i\sigma_1(\xi)}+h^{(2)}(\xi), \tag{9}$$

where $A_k^0(\xi), B_k^0(\xi), C_k^0(\xi), D_k^0(\xi), h^{(2)}(\xi)$ are completely determined functions.

Let $\{\omega_j(\xi)\}_{j=1}^m$ be a fundamental system of solutions of the homogeneous equation

$$\frac{d^m\omega}{\partial\xi^m}+\sum_{k=0}^{m-1}A_k^0(\xi)\frac{d^k\omega}{\partial\xi^k}=0,$$

$V(\xi)$ the Wronskian of the system $\{\omega_j(\xi)\}_{j=1}^m$, and $V_{ml}(\xi)$ the algebraic cofactor of the element in $V(\xi)$ lying at the intersection of the m-th row and the l-th column. Then it can be shown that differential equation (8) is equivalent to the following Volterra-type integral equation with respect to the function $\omega(\xi)$:

$$\omega(\xi)-\int_a^{\xi}k(\xi',\xi)\overline{\omega(\xi')}\,d\xi'=g(\xi) \qquad (a\leqslant\xi\leqslant b), \tag{10}$$

where

$$k(\xi',\xi)=\sum_{k=0}^{m-1}(-1)^{k+1}\frac{\partial^k}{\partial\xi'^k}\sum_{l=1}^{m}V_{ml}(\xi')\,\omega_l(\xi)\,\frac{B_k^0(\xi')}{V(\xi')},$$

$$g(\xi)=\sum_{l=1}^{m}\omega_l(\xi)\int_a^\xi\frac{V_{ml}(\xi')}{V(\xi')}\,f(\xi')\,d\xi'+\sum_{l=1}^{m}\left(C_l+\sum_{k=1}^{m-1}kB_k^0(a)\overline{C_k}\right)\omega_l(\xi),\quad(11)$$

and C_l are complex constants: $C_l(d^{l-1}\omega/d\xi^{l-1})_{\xi=a}$.

Solving equation (10) by the method of successive approximations, we obtain for $\omega(\xi)$ the expression

$$\omega(\xi)=\sum_{j=1}^{n}\int_a^\xi\left(A_j^{(1)}(\xi',\xi)\left(\frac{\partial^j\psi}{\partial\xi^j}\right)_{\zeta=\xi'+i\sigma_1(\xi')}+B_j^{(1)}(\xi',\xi)\left(\frac{\overline{\partial^j\psi}}{\partial\bar{\xi}^j}\right)_{\zeta=\xi'+i\sigma_1(\xi')}\right)d\xi'+$$

$$+\sum_{j=1}^{m}\left(e_j^0(\xi)C_j+d_j^0(\xi)\overline{C_j}\right)+h(\xi),\qquad(12)$$

where the functions $A_j^{(1)}(\xi',\xi),B_j^{(1)}(\xi',\xi),e_j^0(\xi),d_j^0(\xi),h(\xi)$ are completely determined; moreover, $h(\xi)\equiv0$ when $h^{(1)}(\xi)\equiv0$.

Thus, using boundary condition (6), we have managed to express the function $\omega(\xi)$ in terms of the boundary values of the functions $\partial^j\psi/\partial\zeta^j$ and the constants $C_1,C_2,\dots,C_m$. But since $\psi(\zeta)$ is a regular solution of the elliptic equation $\partial\psi/\partial\bar\zeta-q(\zeta)\partial\psi/\partial\zeta=C_1(\zeta)\psi+D_1(\zeta)\overline{\psi}$, the representation (1) holds:

$$\psi(\zeta)=\psi_0(\zeta)+\iint_G\left(R_1(t,\zeta)\psi_0(t)+R_2(t,\zeta)\overline{\psi_0(t)}\right)dG_t,\qquad(13)$$

where $\psi_0(\zeta)$ is the general solution of class $C_n(\overline{G})$ of the Beltrami equation $\partial\psi/\partial\bar\zeta-q(\zeta)\partial\psi/\partial\zeta=0$. Making use now of the integral representa-

by I. N. Vekua (see ([2]), p. 275) for the function $\psi_0(\zeta)$:

$$\psi_0(\zeta)=\int_\gamma\mu(t)\left(1-\frac{W(\zeta)}{W(t)}\right)^{n-1}\ln\left(1-\frac{W(\zeta)}{W(t)}\right)ds+\int_\gamma\mu(t)\,ds+i\beta_0,\quad(14)$$

where $W=W(\zeta)$ is a homeomorphism of class C_ν^n of the Beltrami equation (1), $\mu(t)$ is an unknown real function, Hölder-continuous on γ, and β_0 is a real constant; taking into account (4), (12), (13), we find that the boundary conditions (5), (7) are equivalent to the following singular integro-functional equation (cf. ([4])):

$$K(\mu) \equiv \operatorname{Re} A^*(t_0) \cdot \mu(t_0) + \operatorname{Re} B^*(t_0) \cdot \mu[t(\alpha(s_0))] + \frac{1}{\pi} \operatorname{Im} A^*(t_0) \cdot \int_\gamma \frac{\mu(t)\, dt}{t - t_0}$$

$$- \frac{1}{\pi} \operatorname{Im} B^*(t_0) \cdot \int_\gamma \frac{\mu[t(\alpha(s))]\, dt}{t - t_0} + T(\mu) = \sum_{j=0}^{2m} \gamma_j(t_0)\beta_j + h^*(t_0), \quad t_0 \in \gamma, \tag{15}$$

where $\alpha(s)$ is a real function of class C^1_ν establishing a homeomorphism between the parts γ_1, $\gamma - \gamma_1$ of the contour γ, determined from the equation $\xi[\alpha(s)] = \bar{\xi}(s)$; $T(\mu)$ is a completely continuous operator; β_j are real constants,

$$A^*(t) = \frac{(-1)^n (n-1)! \pi i\, a^*(t)}{(W(t))^{n-1} \theta(t)} \left(\frac{\partial W}{\partial t} \right)^{n-1}; \qquad a^*(t) = \begin{cases} i\Delta(t), & t \in \gamma_1, \\ b_n^{(0)}(t), & t \in \gamma - \gamma_1; \end{cases}$$

$$B^*(t) = \frac{(-1)^n (n-1)! \pi i\, b^*(t)}{(W[t(\alpha(s))])^{n-1} \theta[t(\alpha(s))]} \left(\frac{\partial W}{\partial t} \right)^{n-1}_{t = t(\alpha(s))}; \qquad b^*(t) = \begin{cases} 0, & t \in \gamma_1, \\ b_*(t), & t \in \gamma - \gamma_1, \end{cases}$$

$$b_*(t) = \left(a_m^{(0)}(t) \overline{\Gamma_1^0(t)} + a_m^{(0)}(t) \overline{\Gamma_2^0(t)} \right) \overline{\Gamma_1[t(\alpha(s))]}\, \overline{D_n[t(\alpha(s))]},$$

$$D_n(t) = -\frac{1}{\Delta(i)} \left(\overline{b_n^{(0)}(t)}\, b_{12}(t) - \overline{b_n^{(1)}(t)}\, b_{02}(t) + \overline{b_n^{(2)}(t)}\, b_{01}(t) \right),$$

$$b_{kl}(t) = \overline{a_m^{(k)}(t)}\, b_n^{(l)}(t) - \overline{b_n^{(k)}(t)}\, a_m^{(l)}(t).$$

Taking into account the inequalities (N) and $b_n^{(0)}(t) \neq 0$, $t \in \gamma - \gamma_1$, on the basis of the results on equations of the form (15) obtained in ([3]) (see also ([4])), we are convinced of the validity of the theorem.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.