SOME EXPERIMENTAL CONSEQUENCES OF THE ANALYTICITY OF THE FORM FACTOR

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Abstract

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SOME EXPERIMENTAL CONSEQUENCES OF THE ANALYTICITY OF THE FORM FACTOR

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Consider a certain form factor $F(t)$ and suppose that it is an analytic function in the complex t-plane with a cut on the real axis from $t = 4m_\pi^2$ to ∞. In local field theory $F(t)$, as $t \to \infty$, can grow only more slowly than any linear exponential of $\sqrt{t}$:

$$|F(t)| \leqslant \exp[\varepsilon |t|^{1/2}], \qquad t \to \infty \tag{1}$$

for any $\varepsilon > 0$ ([1,2]). In a number of works ([3–8]) it was shown that from the analytic properties of the form factor one can obtain a number of experimentally verifiable consequences. In the present work we consider some other consequences.

1. Let us first note that as $t \to +\infty$ (in the physical region of the annihilation channel) $F(t)$ cannot decrease faster than $\exp[-\operatorname{const}\sqrt{t}]$. More precisely, there exists some sequence of points $t_n \to +\infty$ such that on it the inequality

$$|F(t_n)| \geqslant \operatorname{const} \cdot \exp[-a\sqrt{t_n}], \qquad a > 0, \qquad t_n \to +\infty \tag{2}$$

holds.

In order to prove this assertion, it is sufficient to make the change of variables $t = z^2$ and then apply to the function $f(z) \equiv F(t)$, analytic in the upper half-plane z, the following theorem.

Theorem 1. Let the function $f(z)$ be analytic in the upper half-plane $\operatorname{Im} z > 0$ and bounded at every finite point of the real axis. If $f(z)$ grows no faster than some linear exponential in the upper half-plane,

$$|f(z)| \leqslant \operatorname{const} \cdot \exp[b|z|], \qquad b > 0, \qquad z \to \infty, \quad \operatorname{Im} z > 0,$$

and decreases exponentially on the real axis:

$$|f(z)| \leqslant \mathrm{const} \cdot \exp[-c|z|], \qquad c > 0, \qquad z \to \pm\infty,$$

then $f(z) \equiv 0$.

A similar theorem was proved in ([9]) (see Theorem 5.8) for functions analytic also on the real axis. The theorem formulated here can be proved analogously if, instead of the maximum principle (see ([9]), Theorem 5.1), one applies the generalized maximum principle (see ([10]), Chapter VI, § 5).

Inequality (2) can also be obtained in a more general case, when the function $F(t)$ is analytic only outside some finite region of the t-plane with a cut. For this purpose it is sufficient to apply a suitable conformal mapping and use the theorem formulated above.

2. If we further suppose that $F(t)$ as $t \to -\infty$ and $|F(t)|$ as $t \to +\infty$ do not oscillate, but have some regular behavior (which can be checked experimentally), then we can obtain stronger results. Applying the Phragmén–Lindelöf theorem in the general form-

level, given, for example, in works ([5,11]), one can show that if

$$F(t) \to a/|t|^n, \qquad t \to -\infty,$$

then

$$|F(t)| \gtrsim |a|/t^n, \qquad t \to +\infty;$$

if

$$F(t) \to a \exp[-b|t|^\alpha], \qquad t \to -\infty, \qquad b > 0, \qquad 0 < \alpha \leq \tfrac{1}{2},$$

then

$$|F(t)| \gtrsim |a| \exp[-b \sin \pi\alpha\, t^\alpha], \qquad t \to +\infty.$$

In particular, if the interaction is minimal in the sense of Martin ([3]) (see also ([12])), i.e.

$$F(t) \to a \exp[-b\sqrt{|t|}], \qquad t \to -\infty, \qquad b > 0,$$

then

$$|F(t)| \gtrsim |a|, \qquad t \to +\infty.$$

In the case when $|F(t)|$ oscillates as $t \to +\infty$, there must exist a sequence of points $t_n \to +\infty$ on which one of the above inequalities holds, provided the corresponding condition on $F(t)$ is satisfied as $t \to -\infty$.

3. Let us further assume that $F(t)$ is bounded on the cut:

$$|F(t)| \leq M, \qquad t \geq 4m_\pi^2, \tag{3}$$

and show that for the values of $|F(t)|$ in the region $t < 0$ there exists a certain lower bound. For this purpose we first make the change of variables $w = [t/4m_\pi^2 + \alpha]^{1/2}$, where α is a positive sufficiently large number, and set $F(t) \equiv g(w)$. The t-plane with a cut is transformed into the upper half-plane w. Since $g(w)$ takes real values on the interval $-\sqrt{1+\alpha} < w < \sqrt{1+\alpha}$, by the Riemann–Schwarz symmetry principle it can be analytically continued into the lower half-plane. Thus, $g(w)$ is an analytic function in the w-plane with cuts $(-\infty, -\sqrt{1+\alpha})$ and $(\sqrt{1+\alpha}, \infty)$.

By means of the conformal mapping

$$\xi = \frac{\sqrt{1+\alpha}}{w}\left[\sqrt{1+\alpha} - \sqrt{1+\alpha-w^2}\right]$$

we transform the w-plane with cuts into the circle C of radius $\sqrt{1+\alpha}$ and with center at zero. The point $w = \sqrt{\alpha}$ is transformed into the point $\xi = a$,

$$a = \frac{\sqrt{1+\alpha}}{\sqrt{\alpha}}\left(\sqrt{1+\alpha} - 1\right), \tag{4}$$

and the points $w = \pm\sqrt{\alpha-\gamma}$, where $\gamma < \alpha$ is a certain fixed positive number, are transformed into the points $\xi = \pm b$,

$$b = \frac{\sqrt{1+\alpha}}{\sqrt{\alpha-\gamma}}\left(\sqrt{1+\alpha} - \sqrt{1+\gamma}\right). \tag{5}$$

The circle C completely contains the ellipse E with foci at the points $\xi = \pm b$ and with major semiaxis $\sqrt{1+\alpha}$. By means of the conformal mapping

$$\eta = \frac{1}{b}\left[\xi + \sqrt{\xi^2 - b^2}\right]$$

we transform this ellipse E into an annulus with inner radius 1 and outer radius R:

$$R = \frac{1}{b}\left[\sqrt{1+\alpha} - \sqrt{1+\alpha-b^2}\right], \tag{6}$$

following Cerulus and Martin [13]. The point $\xi = a$ (i.e., $w = \sqrt{a}$, $t = 0$) is transformed into the point $\eta = r$

$$r = \frac{1}{b}\left[a + \sqrt{a^2 - b^2}\right]. \tag{7}$$

Let $h(\eta) \equiv g(w) \equiv F(t)$. According to the assumption

$$\max_{|\eta|=R} |h(\eta)| \leq M$$

(see formula (3)), while $h(r) = F(0) = 1$. From Hadamard' s theorem on three circles (see [9], Theorem 5.3) it follows that

$$\max_{|\eta|=1} |h(\eta)| = \max_{-\alpha \leq t/4m_\pi^2 \leq -\gamma} |F(t)| \geq \left(\frac{1}{M}\right)^{\frac{\ln r/\ln R}{1-\ln r/\ln R}}.$$

Letting α tend to infinity and using expressions (4)—(7), we obtain *

$$\max_{t \leq -4m_\pi^2 \gamma} |F(t)| \geq \left(\frac{1}{M}\right)^{\Phi(\gamma)}, \tag{8}$$

where

$$\Phi(\gamma) = \frac{\left[1 - (1+\gamma)^{-1/2}\right]^{1/2}}{1 - \left[1 - (1+\gamma)^{-1/2}\right]^{1/2}}. \tag{9}$$

If $F(t)$ decreases monotonically with increasing $|t|$ in the region $t < 0$, then we have

$$F(t) \geq \left(\frac{1}{M}\right)^{\Phi(|t|/4{m_\pi}^2)}. \tag{10}$$

It follows from this inequality that the form factor can decrease by a factor of e in the interval $(-t_e, 0)$ only if t_e satisfies the condition

$$t_e \geq \frac{1}{(1+\ln M)^2 - 1}. \tag{11}$$

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[*] For pions this relation contains only experimentally measurable quantities.

Note: Figure translations are in progress. See original paper for figures.