ON THE STABILITY OF DIFFERENCE SCHEMES IN A COMPLEX HILBERT SPACE

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Abstract

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ON THE STABILITY OF DIFFERENCE SCHEMES IN A COMPLEX HILBERT SPACE

In papers ([1]-[4]), questions in the theory of difference schemes as operator equations in a real Hilbert space were studied. In the present note it is shown that the principal methods and results of papers ([1]-[4]) can be extended to the case of complex spaces. For two-layer schemes with constant operators, necessary and sufficient conditions for stability with respect to initial data in various energy spaces are obtained. Sufficient conditions for the stability of three-layer schemes are also obtained. The isolation of classes of stable schemes makes it possible to carry out regularization and the construction of absolutely stable factorized schemes in a complex Hilbert space. We note that a priori estimates for some difference schemes in a complex Hilbert space were obtained in papers ([5]-[8]).

1. Let us explain the notation: $\{H_h\}$ is a set of complex Hilbert spaces depending on the parameter h; h is an element of some normed space with norm $|h|$; $\omega_\tau = \{t_n = n\tau,\ n = 0, 1, \dots, n_0\}$ is a grid on the interval $0 \le t \le t_0$ with step $\tau = t_0/n_0$; $y(t) = y_{h,\tau}(t)$ is a function of the real discrete argument $t = t_n$ with values in H_h. The indices h, τ will often be omitted below.

In H_h a scalar product $(y, v) = \overline{(v, y)}$ and the norm $\|y\| = \sqrt{(y, y)}$ are defined. Any linear operator A can be represented as the sum

$$A = A_0 + iA_1, \qquad A_0 = \operatorname{Re} A = \frac{1}{2}(A^* + A), \qquad A_1 = \operatorname{Im} A =$$

$$= \frac{i}{2}(A^* - A).$$

An operator A acting in the complex Hilbert space H is called positive, $A > 0$, if $A = A^*$ and $(Ax, x) > 0$ for all $x \in H,\ x \ne 0$; nonnegative, $A \ge 0$, if $A = A^*$ and $(Ax, x) \ge 0$ for all $x \in H$.

If $A > 0$, then one may consider the Hilbert space H_A of elements $y, v \in H$ with scalar product $(y, v)_A = (Ay, v)$ and norm $\|y\|_A = \sqrt{(Ay, y)}$.

We consider two-layer difference schemes (see ([1]))

$$By_t + Ay = 0, \qquad y(0) = y_0, \tag{1}$$

where A and B are linear operators on H_h, $y_0 \in H_h$ is a prescribed element, $y = y_n = y(t_n)$, $y_t = (y_{n+1} - y_n)/\tau$, $n = 0, 1, \dots, n_0 - 1$.

2. Following ([2]), we shall say that scheme (1) is stable with respect to initial data in H_A, where $A > 0$ is a constant linear operator on H_h, if there exists a real number c_0, independent of τ and h, such that, for sufficiently small τ and $|h|$, for the solution of problem (1) with arbitrary initial data $y_0 \in H_h$, the estimate

$$\|y_n\|_A \leq e^{c_0 t_n} \|y_0\|_A = \rho^n \|y_0\|_A, \qquad t_n = \tau n, \quad n = 1, 2, \dots, n_0. \tag{2}$$

holds. Scheme (1) is absolutely stable if (2) is satisfied for arbitrary $\tau > 0$ and $|h| > 0$.

In this note we restrict ourselves to the study of stability with constant operators A and B.

Along with (1), we shall consider the explicit scheme

$$x_t + Cx = 0, \qquad x(0) = x_0 \tag{3}$$

or, in another notation,

$$x_{n+1} = Sx_n, \qquad n = 0, 1, \dots, n_0 - 1, \qquad S = E - \tau C. \tag{4}$$

The following lemma makes it possible to reduce the study of the stability of scheme (1) to the study of the stability of the explicit scheme (3).

Lemma 1. Suppose that in (1) A and B are constant (independent of t) operators. Then, if $A > 0$ and B^{-1} exists, the stability in H of scheme (3) with $C = A^{1/2} B^{-1} A^{1/2}$ is equivalent to the stability in H_A of scheme (1). If $B > 0$, then the stability in H of scheme (3) with $C = B^{-1/2} A B^{-1/2}$ is equivalent to the stability in H_B of scheme (1).

We omit the proof of this lemma, in view of the complete analogy with the case of real spaces (see ([2])).

Just as in (2), one can show that, if the operator C is constant, then for the stability in H of scheme (3) it is necessary and sufficient that the estimate

$$\|S\| \leq \rho, \tag{5}$$

hold, where $\rho = e^{c_0 \tau}$ and $S = E - \tau C$.

If in (1) A and B are constant positive operators, then all the theorems from ([2]) remain valid, and therefore we do not formulate here the corresponding results for schemes in a complex space.

3. We prove lemmas that make it possible to study the stability of the explicit scheme (3).

Lemma 2. Let $S = E - \tau C$, where $C = C_0 + iC_1$, $C_0 \geq 0$, and the operator C^{-1} exists. Then the condition

$$(1+\rho)(C^{-1})_0 \geq \tau E, \qquad (C^{-1})_0 = \operatorname{Re} C^{-1}, \tag{6}$$

for $\rho \geq 1$ is sufficient, and for $0 < \rho \leq 1$ is necessary, for the estimate (5) to hold.

Proof. Noting that (6) is equivalent to the condition

$$\tau \|Cx\|^2 \leq (1+\rho)(C_0 x, x), \tag{7}$$

we obtain that, if (6) is satisfied, then for any $x \in H$

$$\|Sx\|^2 = \|x\|^2 - 2\tau(C_0 x, x) + \tau^2 \|Cx\|^2 \leq \|x\|^2 + \tau(\rho - 1)(C_0 x, x). \tag{8}$$

Further, from (7) and from the inequality

$$(C_0 x, x)^2 \leq \|x\|^2 \|Cx\|^2 \tag{9}$$

we obtain the estimate

$$\tau(C_0 x, x) \leq (1+\rho)\|x\|^2,$$

substituting which into (8), we see that for $\rho \geq 1$ and any $x \in H$ we have $\|Sx\|^2 \leq \rho^2 \|x\|^2$, i.e., inequality (5) is valid.

Conversely, if estimate (5) is satisfied, then for any $x \in H$ the inequality

$$(1-\rho^2)\|x\|^2 - 2\tau(C_0 x, x) + \tau^2 \|Cx\|^2 \leq 0 \tag{10}$$

holds, whence, taking (9) into account, we obtain that

$$(1+\rho)\|x\| \geq \tau \|Cx\|. \tag{11}$$

If $\rho \leq 1$, then (7) follows from (10) and (11).

Lemma 3. Let $S = E - \tau C$, where $C = C_0 + iC_1$, $\rho > 0$ is a number. Then the condition

$$\tau C_0 \leq (1+\rho)E \tag{12}$$

is necessary for the estimate (5) to hold.

Proof. Since for any operator $\|\operatorname{Re} S\| \leq \|S\|$, from (5) the estimate follows

$$\|E - \tau C_0\| \leq \|E - \tau C\| \leq \rho.$$

Hence, taking into account the self-adjointness of the operator C_0, we have

$$-\rho E \leq E - \tau C_0 \leq \rho E.$$

From these inequalities we obtain, in particular, condition (12).

4. The theorems formulated below are consequences of Lemmas 1-3 of the present paper, Theorem 1 and Lemma 2 from ([2]).

Theorem 1. Let in scheme (1) A and B be constant operators, $B = B_0 + iB_1$, $B_0 \geq 0$, B^{-1} exist, $A > 0$; $\rho = e^{c_0\tau}$, $c_0 \geq 0$. Then the condition

$$(1+\rho)B_0 \geq \tau A \tag{13}$$

is sufficient, and the condition

$$\tau(B^{-1})_0 \leq (1+\rho)A^{-1} \tag{14}$$

is necessary for stability in H_A of scheme (1). Condition (13) with $\rho = 1$ is necessary and sufficient for stability (with $c_0 = 0$) of scheme (1) in H_A.

Theorem 2. Let in scheme (1) A and B be constant operators, $B > 0$, $A = A_0 + iA_1$, $A_0 \geq 0$, A^{-1} exist and $\rho = e^{c_0\tau}$, $c \geq 0$. Then the condition

$$(1+\rho)(A^{-1})_0 \geq \tau B^{-1} \tag{15}$$

is sufficient, and the condition

$$(1+\rho)B \geq \tau A_0 \tag{16}$$

is necessary for stability in H_B of scheme (1). Condition (15) with $\rho = 1$ is necessary and sufficient for stability (with $c_0 = 0$) of scheme (1) in H_B.

Condition (15) contains inverse operators and therefore is inconvenient for verification. We give two theorems which yield sufficient stability conditions under stronger restrictions on the operators of the difference scheme.

Theorem 3. Let in scheme (1) A and B be constant commuting operators, $B > 0$, $A = A_0 + iA_1$, $A_0 \geq 0$, A a normal operator, $A^*A = AA^*$, and let there exist nonnegative constants c_1 and c_2, independent of h and τ, such that for all $x \in H$ the conditions

$$\sqrt{\tau}\,|(A_1x, x)| \leq c_2(Bx, x), \tag{17}$$

$$(1 + \rho)B \geq \tau A_0, \tag{18}$$

hold, where $\rho = e^{c_1\tau}$. Then scheme (1) is stable in H_B with $c_0 = c_1 + \frac{1}{2}c_2^2$.

Theorem 4. Let in scheme (1) A and B be constant operators, $B > 0$, $A = A_0 + iA_1$, $A \geq 0$, and let there exist nonnegative constants c_1 and c_2, independent of h and τ, such that for all $x \in H$ the conditions

$$|(A_1x, x)| \leq c_2(Bx, x), \tag{19}$$

$$(1 + \rho)B \geq \tau A_0, \tag{20}$$

hold, where $\rho = e^{c_1\tau}$. Then scheme (1) is stable in H_B with $c_0 = c_1 + c_2$.

Without dwelling on the formulation of sufficient stability conditions with respect to the right-hand side and for schemes with variable operators, we note only that in this case estimates analogous to those obtained in ([2],[4]) are also valid.

5. The sufficient stability conditions (13) make it possible to regularize (see ([3])) unstable two-level schemes and to construct absolutely stable factorized schemes.

Consider, for example, the explicit two-level scheme

$$iy_t + Ay = \varphi(t), \qquad A = A_1 + A_2, \qquad A_\alpha > 0, \qquad \alpha = 1, 2. \tag{21}$$

According to Theorem 1, this scheme is stable with respect to the initial data (with $\rho = 1$

in H_A if and only if the condition

$$B_0 \geqslant \frac{1}{2}\tau A. \tag{22}$$

is satisfied.

In the present case $B = iE,\ B_0 = 0,\ B_1 = E$. Consequently, scheme (21) is absolutely unstable. Therefore, instead of (21) one must use the regularized scheme

$$iy_t + \tau R y_t + Ay = \varphi(t), \tag{23}$$

which, according to Theorem 1, is absolutely stable in H_A (with $\rho = 1$) for every R satisfying the condition

$$\operatorname{Re} R = R_0 \geqslant \frac{1}{2}A. \tag{24}$$

Suppose that (24) is satisfied and, moreover,

$$R = R_1 + R_2, \qquad R_\alpha > 0, \qquad \alpha = 1, 2, \qquad R_1 R_2 = R_2 R_1.$$

Then the factorized scheme

$$\widetilde{B} y_t + Ay = \varphi, \qquad \widetilde{B} = -i \prod_{\alpha=1}^{2} (iE + \tau R_\alpha) = iE + \tau R - i\tau^2 R_1 R_2 \tag{25}$$

is absolutely stable in H_A with $\rho = 1$, since

$$\operatorname{Re} \widetilde{B} = \tau R \geqslant \frac{1}{2}\tau A.$$

6. We formulate sufficient conditions for the stability of the three-layer difference scheme

$$By_{\mathring{t}} + \tau^2 R y_{\bar{t}t} + Ay = 0, \qquad y(0) = y_0, \qquad y(\tau) = y_1, \tag{26}$$

where A, B, R are linear operators in H_h, $y = y_n = y(t_n)$,

$$y_t = (y_{n+1} - y_n)/\tau, \qquad y_{\bar{t}} = (y_n - y_{n-1})/\tau, \qquad y_{\mathring{t}} = \frac{1}{2}(y_t + y_{\bar{t}}), \qquad y_{\bar{t}t} = (y_t - y_{\bar{t}})/\tau.$$

Theorem 5. Let, in scheme (26), the operators A, R be constant and self-adjoint, and $B = B_0 + iB_1$. Then, if the conditions

$$B_0 \geqslant 0, \qquad 4R - A \geqslant 0, \qquad A > 0, \tag{27}$$

are satisfied, then for the solution of problem (26) the estimate

$$\|y_n\|_{(1)} \leqslant \|y_1\|_{(1)}, \tag{28}$$

holds, where

$$\|y_n\|^2_{(1)} = \frac{1}{4}(A(y_n + y_{n-1}),\, y_n + y_{n-1}) + ((R - \tfrac{1}{4}A)(y_n - y_{n-1}),\, y_n - y_{n-1}).$$

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REFERENCES

[1] A. A. Samarskii, DAN, 165, No. 5, 1007 (1965).
[2] A. A. Samarskii, DAN, 181, No. 4 (1968).
[3] A. A. Samarskii, Zhurn. vychisl. matem. i matem. fiz., 7, No. 1, 62 (1967).
[4] A. A. Samarskii, Zhurn. vychisl. matem. i matem. fiz., 7, No. 5, 1096 (1967).
[5] O. A. Ladyzhenskaya, Matem. sborn., 39(81), No. 4, 491 (1956).
[6] J. Gunn, SIAM J. on Numerical Analysis, 2, No. 1, 24 (1965).
[7] P. A. Raviart, J. math. pures et appl., 46, No. 1, 11 (1967).
[8] P. D. Lax, L. Nirenberg, Comm. Pure Appl. Math., 19, No. 4, 473 (1966).

Note: Figure translations are in progress. See original paper for figures.