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ON SOME TOPOLOGICAL PROPERTIES OF BANACH STRUCTURES AND ON CONDITIONS FOR THEIR REFLEXIVITY

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We shall adhere to the terminology and notation from the theory of partially ordered spaces adopted in the monograph ([1]). A K-linear is a linear structure. A K-space (K_σ-space) is a K-linear that is conditionally complete (conditionally σ-complete) as a structure. A KN-linear ($K_\sigma N$-space, KN-space) is a K-linear (K_σ-space, K-space) X which is at the same time a normed space in which the norm is monotone, i.e., from $|x| \leq |y|$ it follows that $\|x\| \leq \|y\|$. A KB-linear is a KN-linear complete in norm. A KB-space is a $K_\sigma N$-space X in which the following two conditions are satisfied.

(A). If $x_n \downarrow 0$ in X, then $\|x_n\| \to 0$.

(B). If $0 \leq x_n \uparrow$ and $\sup \|x_n\| < \infty$, then there exists $\sup x_n \in X$.

The conjugate of a Banach space E is denoted by E^*. A subspace in E is its linear closed subset. Banach spaces E and F are called isomorphic if there exists a one-to-one linear continuous mapping of E onto F. We emphasize that the terms subspace, isomorphism, conjugate space are used in this paper only in the sense of the theory of normed spaces. By c_0, l^1, m we denote the usual Banach spaces of numerical sequences. The symbol $m(T)$ denotes the Banach space of all bounded functions on the set T with the uniform norm.

The following theorem of Nakano–Makarov ([2]) is well known: if $\|\cdot\|_1$ and $\|\cdot\|_2$ are two monotone Banach norms on some K-linear X, then they are equivalent.

This theorem shows that the partial ordering in a KB-linear uniquely determines its Banach topology. The converse question naturally arises: to what extent does the topology in a KB-linear determine the properties of its partial ordering? Let us first recall two known results in this direction.

Theorem 1. For any KB-linear X the following conditions are equivalent: (1) X is a KB-space. (2) X is weakly sequentially complete. (3) In X there is no subspace isomorphic to the space c_0.

The equivalence of (1) and (2) was proved by Ogasawara ([3]), and the equivalence of (2) and (3) is contained in the author' s paper ([4]).

Theorem 2. For a norm-complete $K_\sigma N$-space X the following assertions are equivalent: (1) Condition (A) is satisfied in X. (2) Condition (u), introduced by A. Pelczynski ([5]), is satisfied in X, i.e., for every weakly fundamental sequence $\{x_n\}$ in X there exists a sequence $\{y_n\}$ such that for every $f \in X^*$

$$\sum_{n=1}^{\infty} |f(y_n)| < \infty, \qquad \lim_{n\to\infty} f(x_n) = \sum_{n=1}^{\infty} f(y_n).$$

(3) In X there are no subspaces isomorphic to the space m. (4) In X there are no subspaces isomorphic to the known space P of James (see, for example, ([7]), p. 123).

This theorem was proved by the author ([5]).

Let us now recall the following definition (see ([1]), p. 173).

Definition 1. A K-lineal X is called a K-lineal of countable type if every bounded subset of pairwise disjoint elements of it, different from 0, is at most countable.

Let now E be an arbitrary normed space. Consider the following two properties, each of which we shall also call countability of type.

Definition 2. We shall call E a space of countable type if in E there is no subspace isomorphic to the space $m(T)$, where $\overline{\overline{T}} = \aleph_1$.

Definition 3. We shall call E a space of countable type if there exists a total set of functionals $\mathfrak{M} \subset E^*$ such that for every $x \in E$ the set $\{f \in \mathfrak{M} : f(x) \neq 0\}$ is at most countable.

We note that Definition 1 is applicable to an arbitrary K-lineal, while Definitions 2 and 3 are applicable to an arbitrary normed space. If, however, X is a KN-lineal, then one may speak of its countability of type in any of the three indicated senses.

Theorem 3. *Let X be a norm-complete KN-space with a sufficient set of fully linear functionals. Then (assuming the validity of the continuum hypothesis) all three definitions of countability of type are equivalent for X.*

Thus, with the aid of the continuum hypothesis it is possible to show that in norm-complete KN-spaces with a sufficient set of fully linear functionals, countability of type in the usual sense of the theory of semi-ordered spaces is equivalent to certain of their topological properties.

The scheme of the proof of Theorem 3 is as follows: without the continuum hypothesis, from countability of type in the sense of Definition 2 one derives countability of type in the sense of Definition 1; then from countability of type in the sense of Definition 1 one derives countability of type in the sense of

Definition 3. Finally, from countability of type in the sense of Definition 3, with the aid of the continuum hypothesis, one derives countability of type in the sense of Definition 2. In the proof there are used, in particular, certain results of M. Day ([8,9]) and the following lemma, in which the validity of the continuum hypothesis is not assumed.

Lemma. *The space $m(T)$ is not a space of countable type in the sense of Definition 3 if T has the cardinality of the continuum.*

Remark. It is not difficult to give an example of a norm-complete $K_\sigma N$-space with a sufficient set of fully linear functionals which is of countable type in the sense of Definition 3, but is not such in the sense of Definition 1.

It is known (Eberlein' s theorem; see, for example, ([7])) that in an arbitrary Banach space E the weak sequential compactness of a bounded weakly closed set is equivalent to weak compactness. At the same time the unit ball of the space E^* is always weakly* compact, but, generally speaking, is not sequentially compact in this topology. In this connection we give the following theorem, which (under the assumption of the validity of the continuum hypothesis) gives a criterion for the weak* sequential compactness of the unit ball of the space conjugate to an arbitrary norm-complete $K_\sigma N$-space.

Theorem 4. *Let X be a norm-complete $K_\sigma N$-space. Then (assuming the validity of the continuum hypothesis) the following assertions are equivalent: (1) The unit ball of the space X^* is weakly$\cdots$**

sequentially compact, i.e., every norm-bounded sequence $\{f_n\} \subset X^*$ contains a subsequence converging in the weak topology $\sigma(X^*, X)$. (2) Condition (A) is fulfilled in X, and the space X^* is a space of countable type in the sense of any one of the three definitions given above.

Remark. The implication (2) $\Rightarrow$ (1) is valid without the continuum hypothesis, if countable type is understood in the sense of Definition 1.

With the aid of the lemma formulated above one can establish a number of criteria for the reflexivity of KB-lineals. In what follows the term **reflexivity** is understood only in the sense of the theory of normed spaces. Moreover, all the subsequent results are proved without the continuum hypothesis.

Theorem 5. For an arbitrary KB-lineal X the following assertions are equivalent. (1) X is reflexive as a Banach space. (2) X^{***} and X^{****} are spaces of countable type. (3) X is a KB-space, and X^{***} is of countable type.

In the formulation of this theorem, countable type is understood in the sense of any of the three definitions given above.

Remark. In criterion (2) the spaces in question are the third and fourth conjugate spaces of X. The question arises what the natural numbers m and n should be so that countable type of the m-th and n-th conjugate spaces of X would be equivalent to the reflexivity of X. It can be shown that for this it is necessary and sufficient that these numbers have different parity and that the inequalities $m \geq 3$, $n \geq 3$ hold. Similarly, in criterion (3) the third conjugate X^{***} may be replaced by the m-th conjugate space of X if and only if m is odd, with $m \geq 3$.

It is useful to compare Theorem 5 with Ogasawara' s well-known reflexivity criterion: a KB-lineal X is reflexive if and only if X and X^* are KB-spaces.

Using certain results of Day ([8,9]), Lindenstrauss ([10]), and Ando ([11]), one can give reflexivity criteria in terms of rotundity and smoothness of the unit balls (for definitions of these notions see ([7]), p. 187).

Theorem 6. For an arbitrary KB-lineal X, reflexivity is equivalent to each of the following assertions. (1) X^{***} and X^{*****} are isomorphic to Banach spaces with rotund unit balls. (2) X^* and X^{***} are isomorphic to spaces with smooth unit balls. (3) X^* is isomorphic to a space with a smooth unit ball, and X^{***} is isomorphic to a space with a rotund unit ball.

Remark. The well-known Banach space of R. C. James (see ([7]), p. 123) satisfies all these criteria, but is not reflexive. The reason is that this James space is not isomorphic to any KB-lineal.

For completeness we recall one more criterion for the reflexivity of a KB-lineal X, established earlier by the author ([4]): in X there are no subspaces isomorphic to c_0 or l^1.

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REFERENCES

1. B. Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces*, Moscow, 1961.
2. B. M. Makarov, DAN, 107, 17 (1956).
3. T. Ogasawara, J. Sci. Hiroshima Univ., Ser. A, 12, 37 (1942); 13, 41 (1944).
4. G. Ya. Lozanovskii, *Functional Analysis and Its Applications*, 1, no. 3, 92 (1967).
5. G. Ya. Lozanovskii, *Siberian Mathematical Journal*, 10, no. 1 (1969).
6. A. Pełczyński, Bull. Acad. Polon. Sci., Ser. math., astr. et phys., 6, no. 4, 251 (1958).

7. M. M. Day, *Normed Linear Spaces*, Moscow, 1961.

8. M. M. Day, Trans. Am. Math. Soc., 78, 516 (1955).

9. M. M. Day, Proc. Am. Math. Soc., 8, 415 (1957).

10. J. Lindenstrauss, Bull. Am. Math. Soc., 72, no. 6, 967 (1966).

11. T. Ando, Proc. Japan Acad., 33, 429 (1957).

Note: Figure translations are in progress. See original paper for figures.