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ON EXTREMALLY DISCONNECTED BICOMPACTA OF π-WEIGHT CONTINUUM

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A. V. Arhangel' skii, in paper (^{1}), posed the question: is it true that the absolute of an arbitrary bicompactum of weight $\mathfrak{c}$ ($\mathfrak{c} = \exp \aleph_0 = 2^{\aleph_0}$) is topologically homogeneous? In the present note an affirmative answer to this question is given (Corollary 2). Recall that a space X is called extremally disconnected if the closure of every open $U \subset X$ is again open. A system $\mathfrak{B} = \{U_\alpha\}$ of open sets is called a π-base of X (^{2}) if for every open $V \subset X$ there exists $U_\alpha \in \mathfrak{B}$ such that $U_\alpha \subset V$. The minimum of the cardinalities of open π-bases of X is called the π-weight (^{2}) and is denoted by $\pi w X$. The cellularity number cX is the least upper bound of the cardinalities of open disjoint systems of sets in X. If such systems are denoted by $\mathfrak{A}_\alpha$, then $cX = \sup_{\alpha \in A} |\mathfrak{A}_\alpha|$. A cardinal number $\mathfrak{m}$ is called weakly inaccessible (^{3}) if $\mathfrak{m} \neq 0$ and the following two conditions are satisfied: 1) for every set of indices A, $|A| < \mathfrak{m}$, we have $\sum_{\alpha \in A} \mathfrak{n}_\alpha < \mathfrak{m}$ if $\mathfrak{n}_\alpha < \mathfrak{m}$; 2) if $\mathfrak{n} < \mathfrak{m}$, then there exists a cardinal number $\mathfrak{p}$ such that $\mathfrak{n} < \mathfrak{p} < \mathfrak{m}$. The first uncountable weakly inaccessible cardinal number will be denoted by Ξ_1 ($\Xi_0 = \aleph_0$). By condition 2), $\Xi_1 = \lim_{\alpha \in A} \mathfrak{n}_\alpha$ and $\mathfrak{n}_\alpha < \Xi_1$. The minimal cardinality of a set A such that $\lim_{\alpha \in A} \mathfrak{n}_\alpha = \mathfrak{m}$ is called the cofinal character of $\mathfrak{m}$ and is denoted by $cf(\mathfrak{m})$. If $\mathfrak{m} = \Xi_1$, then $cf(\mathfrak{m}) = \mathfrak{m}$; otherwise, if $\mathfrak{m}$ is limit, uncountable, and $\mathfrak{m} < \Xi_1$, then $cf(\mathfrak{m}) < \mathfrak{m}$. The logarithm of a cardinal number $\mathfrak{n} > 0$ is the least cardinal number $\mathfrak{t} = \log \mathfrak{n}$ such that $2^{\mathfrak{t}} \geq \mathfrak{n}$.

Lemma 1. *Let X be a regular space and $cX = \tau$, with $\tau < \Xi_1$; then in X there exists a system $\mathfrak{B}$ of cardinality τ, consisting of disjoint open subsets of X. In other words, in this case cX attains its least upper bound.*

Proof. If $\tau = \aleph_{\mathfrak{p}+1}$, then it is not hard to show that the assertion of the lemma is always satisfied in this case. Let $\tau = \lim_{\alpha \in A} \mathfrak{n}_\alpha$ be a limit cardinal number. Two mutually exclusive cases are possible:

1) There exists an open $V \subset X$ such that for every open $U \subset V$ we have $cU = cV = \tau$.

2) For every open $V \subset X$ there exists an open $U \subset V$ such that $cU < cV = \tau$.

1st case. In this case we shall construct a system of cardinality τ of open disjoint subsets in V. Since $cV = \tau = \lim_{\alpha\in A} \mathfrak{n}_\alpha$ and $\tau < \Xi_1$, we have $cf(\tau) < \tau$, i.e. $|A| < \tau$*. Consequently, there exists a system $\mathfrak{B}$ of disjoint open sets in V such that $|A| \leq |\mathfrak{B}| \leq \tau$. Let $\mathfrak{B}_0 \subset \mathfrak{B}$ and $|\mathfrak{B}_0| = |A|$. Note that for every $U \in \mathfrak{B}_0$ we have $cU = cX$. Let $cX = \lim_{\alpha\in A} \mathfrak{n}_\alpha$. Establish some one-to-one correspondence between $\mathfrak{B}_0$ and the set $\mathfrak{N} = \{\mathfrak{n}_a\}$, $a \in A$. Let $U_a \in \mathfrak{B}_0$ correspond to $\mathfrak{n}_a \in \mathfrak{N}$. Since $cU_a = \tau > \mathfrak{n}_a$, there exists in U_a a system $\mathfrak{B}_a$, consisting of disjoint open subsets of U_a, and $\mathfrak{n}_a \leqslant |\mathfrak{B}_a| \leqslant \tau$. Put $\mathfrak{B} = \bigcup_{a\in A} \mathfrak{B}_a$. Since $|\mathfrak{B}| = \sum_{a\in A} \mathfrak{n}_a = \lim_{a\in A} \mathfrak{n}_a$, the system $\mathfrak{B}$ is the required one.

2nd case. By transfinite induction we construct a system $\mathfrak{B} = \{U_a\}$ of open disjoint subsets of X such that $\bigcup_a U_a = X$ and $cU_a < \tau$ for all $U_a \in \mathfrak{B}$. Let U_1 be some open subset of X such that $cU_1 < \tau$. Suppose that for every $\alpha < \beta$ a sequence $U_1, U_2, \dots, U_\alpha, \dots$, $\alpha < \beta$, of disjoint open subsets of X has been constructed and $cU_\alpha < \tau$ for all $\alpha < \beta$. Put $\Phi = \bigcup_{\alpha<\beta} U_\alpha$. If $\Phi = X$, then the goal has been attained. If $X \setminus \Phi \neq \emptyset$, then by the assumption there exists $U_\beta \subset X \setminus \Phi$ such that $cU_\beta < \tau$. It is clear that $U_\beta \cap U_\alpha = \emptyset$, if $\alpha < \beta$. In no more than τ steps this transfinite process will stop. Thus the required system $\mathfrak{B} = \{U_\alpha\}$ has been constructed. It is asserted that either $|\mathfrak{B}| = \tau$, or $\sup_\alpha(cU_\alpha) = \tau$. Suppose the contrary. Let $|\mathfrak{B}| = \mathfrak{n} < \tau$ and

$$\sup_\alpha(cU_\alpha) = \mathfrak{m} < \tau.$$

Denote $\mathfrak{t} = \max(\mathfrak{m}, \mathfrak{n})$. It is clear that $\mathfrak{t} < \tau$. Let $\mathfrak{A}$ be a system of open disjoint subsets of X such that $\mathfrak{t} < |\mathfrak{A}| \leqslant \tau$. Such a system exists in X, since $cX = \tau$. To each $U_\alpha \in \mathfrak{B}$ assign the set $\mathfrak{M}_\alpha = \{U \in \mathfrak{A},\ U \cap U_\alpha \neq \emptyset\}$. Since the system $\mathfrak{B}$ is dense in X, for each $U \in \mathfrak{A}$ its intersection with some $U_\alpha \in \mathfrak{B}$ is nonempty and, consequently, $\mathfrak{A} = \bigcup_\alpha \mathfrak{M}_\alpha$. On the other hand, $|\mathfrak{M}_\alpha| \leqslant cU_\alpha \leqslant \sup_\alpha(cU_\alpha) \leqslant \mathfrak{t}$ and $|\mathfrak{B}| \leqslant \mathfrak{t}$, consequently, $|\mathfrak{A}| \leqslant \mathfrak{t} \cdot \mathfrak{t} = \mathfrak{t} < |\mathfrak{A}|$—a contradiction. Thus, either $|\mathfrak{B}| = \tau$ or $\sup_\alpha(cU_\alpha) = \tau$. If the first holds, then everything is proved. Let $\sup_\alpha(cU_\alpha) = \tau$ and $cU_\alpha = \mathfrak{n}_\alpha < \tau$. Arrange the set of cardinal numbers in increasing order:

$$\mathfrak{n}_{\alpha(1)} < \mathfrak{n}_{\alpha(2)} < \dots < \mathfrak{n}_{\alpha(\beta)} < \dots < \tau, \qquad \lim_\beta \mathfrak{n}_{\alpha(\beta)} = \tau. \tag{1}$$

Without loss of generality we shall assume that the sequence (1) is strictly increasing. For each cardinal $\mathfrak{n}_{\alpha(\beta+1)}$ of the sequence (1) consider the corresponding set $U_{\alpha(\beta+1)}$. Since $cU_{\alpha(\beta+1)} = \mathfrak{n}_{\alpha(\beta+1)} > \mathfrak{n}_{\alpha(\beta)}$, there exists in $U_{\alpha(\beta+1)}$ a system $\mathfrak{B}_{\beta+1}$ of disjoint open subsets of $U_{\alpha(\beta+1)}$, with $|\mathfrak{B}_{\beta+1}| = \mathfrak{n}_{\alpha(\beta)}$. Put

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* $|A|$ is the cardinality of the set A.

$\mathfrak{B} = \bigcup_{\beta} \mathfrak{B}_{\beta+1}$. Then $|\mathfrak{B}| = \sum_{\beta} \mathfrak{n}_{\alpha(\beta)} = \lim_{\beta} \mathfrak{n}_{\alpha(\beta)} = \tau$. The system $\mathfrak{B}$ is the required one. The lemma is proved.

Example. It is well known ([4],[5]) that, for example, the system of Gödel axioms, if it is consistent, remains consistent after adjoining the axiom asserting the attainability of all uncountable cardinals. Thus, we may postulate both the existence and the nonexistence of the number Ξ_1. Let $\tau = \Xi_1$. In this case we construct a zero-dimensional bicompactum X, for which $cX = \tau$ and in X there does not exist a system of cardinality τ consisting of disjoint open subsets of X. Let $\tau = \lim_{a \in A} \mathfrak{n}_a$, $|A| = \tau$, and let T_a be a discrete space of cardinality $\mathfrak{n}_a$. Denote by $X_a = b_0 T_a$ the one-point compactification of T_a. Then

$$X = \prod_{a \in A} X_a$$

is the Tychonoff product of the bicompacta X_a. Since τ is regular and is a caliber (in the sense of N. A. Shanin) of each X_a, it follows by a theorem of N. A. Shanin ([6]) that τ is a caliber of the bicompactum X. Consequently, in X there is no family of cardinality τ consisting of disjoint open subsets. On the other hand, for each $\mathfrak{n}_a < \tau$ the family $\mathfrak{B}_a = \{\pi_a^{-1} x\}$, $x \in T_a$, where π_a is the projection of X onto X_a, is a family of cardinality $\mathfrak{n}_a$ of disjoint open subsets of X. It is easy

to show that the absolute pX is already an extremally disconnected bicompactum with the properties indicated above.

Lemma 2. *Let X be a regular space, $\pi w X \leq \tau$ and $cX \leq \mathfrak{n}$. Then the cardinality of all canonical closed subsets of X does not exceed $\tau^{\mathfrak{n}}$; in particular $wX \leq \tau^{\mathfrak{n}}$.*

Corollary 1. *If a regular space X satisfies Suslin' s condition and $\tau^{\aleph_0} = \tau$, and moreover $\pi w X = \tau$, then $wX = \tau$.*

Let T be a discrete space, $|T| \geq \aleph_0$, let βT be the Stone–Čech compactification of T, and let $x \in \beta T \setminus T$. Then the subspace $P = x \cup T$, lying in βT, will be called an atom, and the point $x \in P$ the nucleus of the atom. Two atoms P_1 and P_2 will be called equivalent if P_1 is homeomorphic to P_2. Two atoms P_1 and P_2 will be called weakly equivalent if there exist two equivalent atoms $P_1' \subset P_1$ and $P_2' \subset P_2$. A subspace $T = \{t_\alpha\}$ of a space X will be called strongly discrete in X if there exists a system $\mathfrak{B} = \{U_\alpha\}$ of disjoint open subsets of X such that $U_\alpha \ni t_\alpha$. An atom $P = x \cup T$ will be called regularly situated in X if T is strongly discrete.

Lemma 3. *Let X be a regular extremally disconnected space, with $wX \leq \tau$, $cX \leq \mathfrak{n}$, and $x \in X$. Then the cardinality of the set of all regularly situated pairwise not weakly equivalent atoms with nucleus x does not exceed $\tau^{\mathfrak{n}}$.*

Lemma 4. *Let X be an infinite extremally disconnected bicompactum containing a family $\mathfrak{B} = \{U_\alpha\}$ of cardinality $\mathfrak{n}$, consisting of disjoint open subsets of*

X. *Then* X *contains* $\operatorname{expexp}\mathfrak{n}$ *regularly situated pairwise not weakly equivalent atoms.*

Proof. Let $\mathfrak{B}$ be the given family. Choose in each U_α a point $t_\alpha \in U_\alpha$. Put $T = \bigcup t_\alpha$; then T is strongly discrete. It is asserted that $\overline{T} = \beta T$. For this it is enough to prove that $\overline{T}$ is extremally disconnected. Let U and V be open subsets of $\overline{T}$ and $U \cap V = \emptyset$. Put $T_1 = U \cap T$ and $T_2 = V \cap T$; note that $\overline{T_1} = \overline{U}$, $\overline{T_2} = \overline{V}$, and $T_1 \cap T_2 = \emptyset$. Let

$$W(T_i) = \{U_\alpha \in \mathfrak{B},\ U_\alpha \cap T_i \neq \emptyset\}, \quad i = 1, 2.$$

Then

$$L(T_i) = \bigcup U_\alpha, \quad U_\alpha \in W(T_i), \quad i = 1, 2,$$

are disjoint open sets and $L(T_i) \supset T_i$, $i = 1, 2$. Since X is extremally disconnected, $\overline{L(T_1)} \cap \overline{L(T_2)} = \emptyset$, but $\overline{L(T_i)} \supset \overline{T_i}$, as was required. Thus, $\overline{T} = \beta T$. Let $x \in \beta T \setminus T$; then $P = x \cup T$ is a regularly situated atom. We shall prove that the cardinality of all regularly situated in X pairwise not weakly equivalent atoms P, whose nuclei belong to $\beta T \setminus T$, is equal to $\operatorname{expexp}\mathfrak{n}$. Consider two arbitrary weakly equivalent atoms $P_1 = x_1 \cup T_1$ and $P_2 = x_2 \cup T_2$, $x_1 \neq x_2$, $T_1 \subset T$ and $T_2 \subset T$. We shall prove that to each such pair (P_1, P_2) there corresponds an autohomeomorphism $\varphi : \beta T \to \beta T$. Let $P_1' \subset P_1$, $P_2' \subset P_2$, and let P_2' be homeomorphic to P_2'. If $P_1' = x_1 \cup T_1'$ and $P_2' = x_2 \cup T_2'$, then $T_1 \setminus T_1'$ and $T_2 \setminus T_2'$ are discrete, since P_1' and P_2' are extremally disconnected. Let $f : P_1' \to P_2'$ be a homeomorphism of P_1' onto P_2'; then, extending it to $\beta T_1'$, we obtain a homeomorphism $\tilde{f} : \beta T_1' \to \beta T_2'$. Further, leaving $\overline{T}_0 = T \setminus T_1' = T \setminus T_2'$ fixed, we obtain a homeomorphism $\varphi : \beta T$ onto βT such that $\varphi(x_1) = x_2$. But there are no more such homeomorphisms than

$$|T|^{|T|} = \mathfrak{n}^{\mathfrak{n}} = 2^{\mathfrak{n}}.$$

On the other hand, it is known (7) that

$$|\beta T - T| = \operatorname{expexp}\mathfrak{n}.$$

Consequently, the cardinality of the set of all pairwise not weakly equivalent atoms in X is no less than $\operatorname{expexp}\mathfrak{n}$. The lemma is proved.

Theorem 1. *Let X be an extremally disconnected bicompactum and $\pi w X = \tau$, where $\tau < \Xi_1$. If $cX \geq \log\tau$, then X is not topologically homogeneous.**

Proof. It is clear that $cX \leq \pi w X$, hence $cX = \mathfrak{n} \leq \tau < \Xi_1$. Applying Lemma 1, we obtain that in X there exists a system of cardinality $\mathfrak{n}$, consisting of disjoint open subsets of X. By Lemma 4 in this case—

* wX is the weight of X.

** A space X is called topologically homogeneous if for every pair of points $x, y \in X$ there exists an autohomeomorphism $\varphi : X \to X$ such that $\varphi(x) = y$.

where X contains $\operatorname{expexp}\mathfrak{n}$ regularly situated pairwise not weakly equivalent atoms. On the other hand, by Lemma 2,

$$wX \leq (\pi wX)^{cX} = \tau^{\mathfrak{n}}.$$

Applying Lemma 3, we obtain that the cardinality of the set of all regularly situated pairwise not weakly equivalent atoms with kernel at an arbitrary point $x \in X$ does not exceed

$$(wX)^{cX} \leq (\tau^{\mathfrak{n}})^{\mathfrak{n}} = \tau^{\mathfrak{n}^2} = \tau^{\mathfrak{n}}.$$

Since, by assumption, $\mathfrak{n} \geq \log\tau$, it follows that $2^{\mathfrak{n}} \geq \tau$, and therefore

$$\tau^{\mathfrak{n}} \leq (2^{\mathfrak{n}})^{\mathfrak{n}} = 2^{\mathfrak{n}} = \exp\mathfrak{n} < \operatorname{expexp}\mathfrak{n}.$$

Thus the number of regularly situated not weakly equivalent atoms with kernel at an arbitrary point $x \in X$ is strictly less than the number of all regularly situated pairwise not weakly equivalent atoms in X. Hence it follows directly that in X there exist two points that cannot be carried into one another by a homeomorphism of X onto itself. The theorem is proved.

Corollary 2. *If $\mathfrak{c} < \Xi_1$, then the absolute of an arbitrary compactum of weight $\mathfrak{c}$ is nonhomogeneous.*

Indeed, if X is a compactum, $wX = \mathfrak{c}$, pX is the absolute of X, and $\pi : pX \to X$ is an irreducible perfect mapping of pX onto X. Since π-weight is an invariant of such mappings (2), we have

$$\pi w(pX) = \pi wX \leq wX = \mathfrak{c}.$$

On the other hand, since $w(pX) = \mathfrak{c}$, we have $c(pX) \geq \aleph_0 = \log\mathfrak{c}$. Hence by Theorem 1 we obtain that pX is nonhomogeneous.

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REFERENCES

[1] A. V. Arhangel' skii, DAN, **175**, No. 4, 751 (1967).
[2] V. I. Ponomarev, UMN, **21**, No. 4, 101 (1966).
[3] W. Sierpiński, *Cardinal and Ordinal Numbers*, Warszawa, 1958.
[4] R. Baer, Math. Zs., **29**, 381 (1929).
[5] C. Kuratowski, Ann. de la Soc. Polon de Math., **3**, 116 (1925).
[6] N. A. Shanin, Tr. Matem. inst. im. V. A. Steklova, **24**, 3 (1948).
[7] B. Pospíšil, Ann. Math., **38**, 846 (1937).
[8] Z. Frolík, CMUS, **8**, 4, 757 (1967).

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* This paper had already been written when it became known to me that Z. Frolík had proved (8) that if there are cardinal numbers between $\mathfrak{c}$ and 2, then every infinite extremally disconnected compactum is nonhomogeneous. We note that my assumption $\mathfrak{c} < \Xi_1$ and Frolík’ s assumption are completely different: one may hold while the other does not, and conversely. All this indicates an interesting connection between the problem of topological homogeneity of extremally disconnected compacta and hypotheses of set theory.

Note: Figure translations are in progress. See original paper for figures.

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