ON FACTORIAL s-MAPPINGS

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Abstract

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ON FACTORIAL s-MAPPINGS

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The present note is devoted to the study of factorial s-mappings of spaces with a point-countable base onto spaces of point-countable type. All necessary definitions, as well as some comments, may be found in (^{1}).

Theorem 1*. *Let $f : X \to Y$ be a factorial s-mapping of a space X with a point-countable base onto a T_2-space Y of point-countable type. Then the first axiom of countability holds in the space Y.*

We shall carry out the proof in several stages.

I. If Y' is a set closed in Y, and $X' = f^{-1}(Y')$, then $f|_{X'} : X' \to Y'$ is a factorial mapping; therefore the following lemma, proved by A. V. Arhangel' skii for completely regular spaces and by M. Choban for T_2-spaces, allows us to conduct the proof under the assumption that Y is bicompact, which we shall do.

Lemma (see (^{2})). *If a bicompactum B has countable character in a T_2-space Y, and a point $y \in B \subseteq Y$ has countable character in the subspace B, then the point y has countable character also in the space Y.*

II. Let $m = \{y_k^i,\ i, k = 1, 2, ...\} \subseteq Y$ and let

$$y \in \left[\{y_k^i,\ i = 1, 2, \dots,\ k > l\}\right] \setminus \{y_k^i,\ i = 1, 2, \dots,\ k > l\}$$

for every l. Then from the set m one can choose a sequence of points

$$\{y_{k(n)}^{i(n)},\ n = 1, 2, \dots,\ k(n+1) > k(n)\}$$

such that

$$\left[\{y_{k(n)}^{i(n)}\}\right] = \{y_{k(n)}^{i(n)}\} \cup \{y\}.$$

As the sum of a countable number of separable sets, the set $f^{-1}(m)$ is separable, and therefore, for any point-countable base Δ of the space X, the set Δ^* of those of its elements which intersect $f^{-1}(m)$ is at most countable. Let

$$g = \{Y \setminus [f(\delta)],\ \delta \in \Delta^*,\ y \in [f(\delta)]\}.$$

By construction, g is a countable family of open sets in Y containing the point y. We shall suppose the elements of the family g enumerated, i.e. let $g = \{G_n\}$. We construct by induction a family $\gamma = \{\Gamma_n\}$. As Γ_1 take G_1. Suppose open sets $\Gamma_1, \ldots, \Gamma_{n-1}$ have been constructed. As Γ_n take an arbitrary open set satisfying the relation

$$y \in \Gamma_n \subseteq [\Gamma_n] \subseteq \Gamma_{n-1} \cap G_n.$$

Let

$$B = \bigcap_{n=1}^{\infty} \Gamma_n.$$

Since $[\Gamma_n] \subseteq \Gamma_{n-1}$, it follows that

$$B = \bigcap_{n=1}^{\infty} [\Gamma_n],$$

and consequently B is a bicompactum. The family $\gamma = \{\Gamma_n\}$ is a certain base of neighborhoods** of the bicompactum B: for any open set U containing B, beginning with some number N,

$$[\Gamma_n] \subseteq U,$$

for otherwise $[\Gamma_n] \setminus U$ would be a decreasing sequence of nonempty bicompacta; their intersection would be nonempty and would lie, on the one hand, in $\bigcap_{n=1}^{\infty} [\Gamma_n] = B$, and on the other—in $Y \setminus U$, which contradicts the inclusion $B \subseteq U$.

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* *Note added in proof.* Very recently the author obtained a considerably stronger result.

Theorem 1′**.** *A T_2-space of point-countable type which is a factorial s-image of a space with a point-countable base itself has a point-countable base.*

Theorem 1′ completely covers almost all results known to the author in this direction, and also gives an answer to some of the questions that had remained open.

** In other words, for any open set U containing the set B, there is an element Γ of the family γ such that $B \subseteq \Gamma \subseteq U$.

We construct by induction the sequence $\{y^i_{k(n)},\ n = 1, 2, \ldots\}$. As $y^i_{k(1)}$ take any point of the set $\{y^i_k\}$. Suppose the points $y^i_{k(1)}, \ldots, y^i_{k(n-1)}$ have been chosen. Since, by assumption, $y \in [\{y^i_k;\ k > k(n-1)\}]$, it follows that $\Gamma_n \cap \{y^i_k,\ k > k(n-1)\} \neq \emptyset$. As $y^i_{k(n)}$ take any point of the set $\Gamma_n \cap \{y^i_k,\ k > k(n-1)\}$. Since every neighborhood of the set B contains all the points $y^i_{k(n)}$, beginning with some one of them, there are in the bicompactum B (and only there—since a closed set and a point in a bicompactum are separable) points that are limit points for this sequence. We shall show that y is the unique limit point. For this it suffices to show that the set

$$f^{-1}(\{y^i_{k(n)}\} \cup \{y\}) = Q$$

is closed. Suppose this is not so. Then there exists a point $x' \in [Q] \setminus Q$. Let $y' = f(x') \neq y$. Since Y, by assumption (see I), is a bicompactum, there exists an open set $V \ni y'$ such that $y \in [V]$. Take an element δ of the base Δ with $x' \in \delta \subseteq f^{-1}(V)$. Since $x' \in [Q] \subseteq [f^{-1}(m)]$, we have $\delta \in \Delta^*$. Moreover, $f(\delta) \subseteq [f(\delta)] \subseteq [V]$ and $y' \in [V]$; therefore $Y \setminus [f(\delta)]$ is an element of the family g that does not contain the point y'. Hence $y' \in B$. But, as was already noted, the sequence $\{y^i_{k(n)}\}$ has no limit points outside B. Thus the supposition that there exists a point $x' \in [Q] \setminus Q$ has led us to a contradiction and, consequently, y is indeed the unique limit point of the sequence $\{y^i_{k(n)}\}$, as required.

III. We shall show that the mapping f is pseudo-open.

Suppose this is not so. Then there exists an open set $V \subseteq X$ such that $\operatorname{Int} f(V)$ does not contain all points whose preimages lie entirely in V. Consider $U = \operatorname{Int} f^{-1} f(V)$. Since, obviously, $V \subseteq U$ and $f(V) = f(U)$, the set $\operatorname{Int} f(U) = \operatorname{Int} f(V)$ also does not contain all points whose preimages lie entirely in U. It follows that there exist points

$$\begin{aligned}
&y^i_k \in Y \quad (i, k = 1, 2, \ldots), \qquad f^{-1}(y^i_k) \subseteq X \setminus U,\\
&y_k \in Y,\\
&x^i_k \in f^{-1}(y^i_k), \qquad x_k \in f^{-1}(y_k),\\
&x^i_k \to x_k \quad \ast \ \text{as } i \to \infty,\\
&x'_k \in f^{-1}(y_k),\\
&y \in Y, \qquad f^{-1}(y) \subseteq U,\\
&x'_k \to x \in f^{-1}(y).
\end{aligned}$$

Indeed, otherwise we would have that $f^{-1} f(X \setminus U)$ is a closed set, as is

$$f(f^{-1} f(X \setminus U)) = f(X \setminus U),$$

which contains not a single point whose preimage lies entirely in U. Then all such points would lie in the open set

$$Y \setminus f(X \setminus U) \subseteq f(U) = f(V),$$

which contradicts our supposition.

By II, from the set $\{y_k^i\}$ we can choose a sequence $\{y_{k(n)}^i,\ n = 1, 2, \ldots\}$ such that

$$[\{y_{k(n)}^i\}] = \{y_{k(n)}^i\} \cup \{y\}.$$

It follows that the set $f^{-1}(\{y_{k(n)}^i\})$ has no limit points not belonging to it and lying outside $f^{-1}(y)$. But since $f^{-1}(\{y_{k(n)}^i\}) \subset X \setminus U$, where the set $X \setminus U$ is closed and

$$f^{-1}(y) \cap (X \setminus U) = \emptyset,$$

the set $f^{-1}(\{y_{k(n)}^i\})$ has no limit points in $f^{-1}(y)$ either. But then the set $\{y_{k(n)}^i\}$ is closed, which contradicts the inclusion $y \in [\{y_{k(n)}^i\}]$. The contradiction obtained proves that the mapping f is pseudo-open.

IV. Let $y \in Y$, and let $\{H_n\}$ be an arbitrary countable covering of the set $f^{-1}(y)$ by open subsets of X. We shall show that from it one can choose a finite number of elements $\{H_1, \ldots, H_n\}$ such that

$$y \in \operatorname{Int} f(H_1 \cup \cdots \cup H_n).$$

Suppose that this is not so; then, since the mapping f is pseudo-open, for any k there exist sequences of points

$$y_k^i \in Y \quad (i = 1, 2, \ldots), \qquad f^{-1}(y_k^i) \cap (H_1 \cup \cdots \cup H_k) \neq \emptyset,$$

$$x_k^i \in f^{-1}(y_k^i), \qquad x_k^i \to x_k \in f^{-1}(y).$$

In accordance with II, choose from the set $\{y_k^i\}$ a sequence $\{y_{k(n)}^{i(n)}\}$ such that

$$[\{y_{k(n)}^{i(n)}\}] = \{y_{k(n)}^{i(n)}\} \cup \{y\}.$$

Take an arbitrary $x \in f^{-1}(y)$; then, for some N, $x \in H_N$, and H_N is a neighborhood of the point x which does not meet the sets $f^{-1}(y_{k(n)}^{i(n)})$ as soon as $k(n) \geq N$, and consequently

$$x \in [f^{-1}\{y_{k(n)}^{i(n)}\}].$$

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* $x_i \to x$ if every neighborhood of the point x contains all the points x_i, beginning with some one of them.

By II the set $f^{-1}(\{y_{k(n)}^{i(n)}\})$ has no limit points outside $f^{-1}(y)$, and therefore the set $f^{-1}(\{y_{k(n)}^{i(n)}\})$ is closed. Consequently, the set

$$\{y_{k(n)}^{i(n)}\} = ff^{-1}(\{y_{k(n)}^{i(n)}\})$$

is closed as well, which contradicts the equality

$$[\{y_{k(n)}^{i(n)}\}] = \{y_{k(n)}^{i(n)}\} \cup \{y\}.$$

The assertion is proved.

V. The set Δ' of elements of the base Δ which meet $f^{-1}(y)$ is at most countable. Therefore the family of all finite subsets of the set Δ' is also at most countable, as is the family

$$\beta = \{\operatorname{Int} f(\delta_1 \cup \dots \cup \delta_n),\ \delta_1, \dots, \delta_n \in \Delta',\ n = 1, 2, \dots\}.$$

If we show that this family is a certain base at the point, then theorem 1 will thereby be proved. Let V be an open subset of Y containing the point y. For each point $x \in f^{-1}(y)$ there is an element δ of the base Δ such that

$$x \in \delta \subseteq f^{-1}(V).$$

The set of all such elements is at most countable; they all belong to Δ' and together cover the set $f^{-1}(y)$. By IV, from this family one can choose a finite number of elements $\delta_1, \dots, \delta_n$ such that

$$y \in \operatorname{Int} f(\delta_1 \cup \dots \cup \delta_n),$$

i.e. $\operatorname{Int} f(\delta_1 \cup \dots \cup \delta_n)$ is an element of the family β containing the point y and contained in the set V, as was required.

From the corollaries of the theorem just proved we give one (see (2)).

Theorem 2. Let $f : X \to Y$ be a factor s-mapping of a metric space X onto a regular separable space Y of point-countable type. Then the space Y is metrizable.

In particular:

Corollary. A separable bicompactum which is a factor s-image of a metric space is metrizable.

In conclusion I express my gratitude to my scientific adviser A. V. Arhangel' skii, and also to M. Choban for valuable discussions.

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Note: Figure translations are in progress. See original paper for figures.

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