ON A DIFFERENTIAL GAME OF APPROACH

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Abstract

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The works of many authors have been devoted to conflict problems of the approach of controlled objects at a prescribed moment of time $t = \vartheta$ (see, for example, ([1–8])). In a number of cases these problems are of a more regular character than problems on the minimax or maximin of the time to encounter (see, in this connection, for example, ([9–11])). In the present paper it is shown that the rule of extremal aiming ([11]), based on extremal points of the attainability domains of the pursued and pursuing motions and modified from the standpoint of generalized differential equations in contingencies ([12,13]), ensures a saddle point of the game in the rough case considered in this paper.

Consider the pursuing (y) and pursued (z) motions described by the differential equations

$$\dot{y} = f^{(1)}[t, y] + B^{(1)}[t, y]u \tag{1}$$

$$\dot{z} = f^{(2)}[t, z] + B^{(2)}[t, z]v, \tag{2}$$

where y, z are the phase n-dimensional vectors of the objects; u, v are r-dimensional control vectors, the realizations of which $u[t]$ and $v[t]$ are constrained by

$$u[t] \in \mathcal{U}, \qquad v[t] \in \mathcal{V}, \tag{3}$$

with $\mathcal{U}$ and $\mathcal{V}$ bounded, convex, and closed sets; $f^{(i)}$, $B^{(i)}$ are vector-functions and matrix-functions, respectively, continuous in t and continuously differentiable in y and z; $t = t_0$ is the initial moment, $t = \vartheta$ is the fixed terminal moment of the process. The symbol $\{x\}_m$ will denote the vector composed of the first m components of the vector x, a superscript prime will denote transposition, and the symbol $\|q\|$ the sign of the Euclidean norm. The problem consists in constructing controls u^0, v^0 according to the feedback principle in the form

$$u = u^0(t, y, z), \qquad v = v^0(t, y, z) \tag{4}$$

so that, for any possible initial state $t = \tau,\ y(\tau),\ z(\tau)\ (t_0 \leq \tau < \vartheta)$, the condition

$$\|\{y_{u^0,v}[\vartheta] - z_{u^0,v}[\vartheta]\}_m\| \leq \|\{y_{u^0,v^0}[\vartheta] - z_{u^0,v^0}[\vartheta]\}_m\| \leq \|\{y_{u,v^0}[\vartheta] - z_{u,v^0}[\vartheta]\}_m\|. \tag{5}$$

Here the symbols $y_{u,v}[t]$ and $z_{u,v}[t]$ denote the realizations of the motions y and z, generated by equations (1) and (2) under the chosen controls

$$u = u(t, y, z), \qquad v = v(t, y, z), \tag{6}$$

the number m is given by the conditions of the problem.

Denote by the symbols $G_\varepsilon^{(1)}[\tau, y]$ and $G_\varepsilon^{(2)}[\tau, z]$ $(\varepsilon \geq 0)$ the closed ε-neighborhoods of the attainability domains ([11]) (in the space of the vectors $q = \{y\}_m,\ q = \{z\}_m$) for motions (1) and (2) from the states $y(\tau) = y,\ z(\tau) = z$ to the moment $t = \vartheta$ and under constraints (3). We shall say that the **rough case** takes place if the following conditions are satisfied:

1°. The domains $G_\varepsilon^{(i)}$ are convex.

2°. For all possible τ, y, z and $\varepsilon > 0$, the boundaries of the domains $G_\varepsilon^{(1)}$ and $G_0^{(2)}$ have no more than one common point q, when $G_0^{(2)} \subset G_\varepsilon^{(1)}$.

3°. For each point q lying on the boundary $G_0^{(1)}[\tau, y]$ (or on the boundary $G_0^{(2)}[\tau, z]$) there exists only one, in essence, programmed optimal control $\hat{u}_{y,\tau}(t)_q$ (or $v^0_{z,\tau}(t)_q$), which transfers system (1) (or (2)) from the state $y(\tau) = y(z(\tau) = z)$ to the state $\{y(\vartheta)\}_m = q$ $(\{z(\vartheta)\}_m = q)$.

The generalized extremal control $u = u^0(t, y, z)$ is constructed as follows. Suppose that at some instant $t = \tau$ the quantities $y[\tau] = y^*$ and $z[\tau] = z^*$ have been realized. We construct the attainability domains $G_0^{(2)}[\tau, z^*]$ and $G_{\varepsilon^0}^{(1)}[\tau, y^*]$, where ε^0 is the least number for which

$$G_0^{(2)}[\tau, z^*] \subset G_{\varepsilon^0}^{(1)}[\tau, y^*]. \tag{7}$$

Let first $\varepsilon^0 > 0$ and let p^* be the point of tangency of the boundaries of the domains $G_{\varepsilon^0}^{(1)}$ and $G_0^{(2)}$, and let q^* be the point of the domain $G_0^{(1)}$ nearest to the point p^*. The programmed optimal control $u^0_{y^*,\tau}(t)_{q^*}$ $(\tau \leqslant t < \vartheta)$, which transfers object (1) from the state $y(\tau) = y^*$ to the position $\{y(\vartheta)\}_m = q^*$, at $t = \tau$ satisfies the requirements of the maximum principle [14]

$$\psi'[\tau] B^{(1)}[\tau, y^*]\, u^0_{y^*,\tau}(\tau)_{q^*} = \max_{u \in \mathcal{U}} (\psi'[\tau] B^{(1)}[\tau, y^*] u), \tag{8}$$

where $\psi'(\vartheta) = \{l^*, 0\}$, where $l^*[\tau]$ is an r-dimensional vector orthogonal to the plane supporting $G^{(1)}_{\varepsilon^0}[\tau, y^*]$ at the point p^*. The function $u^0[t, y, z]$ defining the extremal control is regarded as multivalued, and at each point $t = \tau,\ y = y^*,\ z = z^*$ we allow for it all values $u = u^0_{y^*,\tau}(\tau)_{q^*}$ satisfying condition (8). The set of these values will be denoted by the symbol $U^0[\tau, y^*, z^*]$. If $\varepsilon^0 = 0$, then we put $U^0[\tau, y^*, z^*] = \mathcal{U}$. In an analogous way the extremal control $v = v^0(t, y, z) \in V^0(t, y, z)$ is constructed, but now from the condition of aiming at the point $\{z(\vartheta)\}_m = p^*$. By motions (solutions) of systems (1), (2) in the case of multivalued $u = u(t, y, z) \in U[t, y, z]$ and $v = v(t, y, z) \in V[t, y, z]$ we shall mean any pair of absolutely continuous functions $y[t]$ and $z[t]$ satisfying the equations

$$\dot{y}[t] = f^{(1)}[t, y[t]] + B^{(1)}[t, y[t]]u[t], \qquad u[t] \in U[t, y[t], z[t]],$$

$$\dot{z}[t] = f^{(2)}[t, z[t]] + B^{(2)}[t, z[t]]v[t], \qquad v[t] \in V[t, y[t], z[t]],$$

for almost all $t \in [t_0, \vartheta]$. Systems of sets $U[t, y, z] \subset \mathcal{U}$ and $V[t, y, z] \subset \mathcal{V}$ for which system (1), (2) will possess motions (solutions) will be called admissible. The following assertion is valid:

In the rough case, the extremal controls $u^0 \in U^0[t, y, z]$ and $v^0 \in V^0[t, y, z]$ are admissible and ensure the saddle point (5) of the game under consideration. Moreover, $\|\{y_{u^0,v^0}[\vartheta] - z_{u^0,v^0}[\vartheta]\}_m\| = \varepsilon^0[\tau]$.

The existence of motions $y[t]$ and $z[t]$ for $u \in U^0[t, y, z]$ or $v \in V^0[t, y, z]$, i.e. the existence of solutions of the corresponding system of equations (1), (2) in contingencies, is verified by passage to the limit from approximating Euler polygonal lines, which are generated by the controls $u = u_\delta[t]$ (or $v = v_\delta[t]$) changing their values in connection with the developing position $\{\tau, y[\tau], z[\tau]\}$ only at discrete instants of time $\tau = \tau_k$ ($\tau_{k+1} - \tau_k = \delta > 0$). The minimax property of the control $u = u^0$ follows from the condition that for $u = u^0 \in U^0[t, y, z]$ the quantity $\varepsilon^0[t] > 0$ for $t \geqslant \tau$ does not increase. This follows from the estimate of this quantity, determined by the equality

$$\min_{\|l\|=1} \left(\rho^{(1)}[l, t, y[t]] - \rho^{(2)}[l, t, z[t]] + \varepsilon^0[t]\right) = 0, \tag{9}$$

where $\rho^{(i)}$ are the support functions for the domains $G^{(i)}_0$, respectively.

The maximin property of the control $v = v^0$ follows from the condition that, for $v = v^0 \in V^0[t, y, z]$, the quantity $\varepsilon^0[t] > 0$, on the contrary, does not decrease with increasing time. It is important here that, in the rough case under consideration, the vector $l = l^*[t]$, which ensures the minimum in (9) and has thereby become a supporting vector to the regions $G^{(1)}_{\varepsilon^0}[t, y[t]]$ and $G^{(2)}_0[t, z[t]]$ at the point $p^*[t]$, varies continuously with the time t when $u = u^0$ or $v = v^0$.

Together with the vector $l^*[t]$, the vector $\psi[t]$ from condition (8) also varies continuously.

Remark. The motions $y_{u,v}[t]$ and $z_{u,v}[t]$ for admissible $u \in U[t, y, z]$ and $v \in V[t, y, z]$, generally speaking, are not unique. Therefore the assertion made has the meaning that the left (and right) condition (5) is fulfilled for any pair of two motions $\{y_{u^0,v}, z_{u^0,v}\}$ and $\{y_{u,v^0}, z_{u,v^0}\}$ (or for $\{y_{u^0,v^0}, z_{u^0,v^0}\}$ and $\{y_{u,v^0}, z_{u,v^0}\}$, respectively).

The motions $y_{u,v}[t]$ and $z_{u,v}[t]$ are in fact realized, generally speaking, as sliding regimes, and the description given above is their formalization in the scheme of solutions of generalized differential equations in contingencies. A substantive construction of these solutions is obtained in limiting form from the discrete scheme described in paper (11). In this case condition (5) takes the form

$$\lim_{\delta\to 0}\sup_{v}(\sup \|\{y_{u^0_\delta,v}[\vartheta] - z_{u^0_\delta,v}[\vartheta]\}_m\|) =$$

$$= \min_{v_\delta}\left(\lim_{\delta\to 0}\sup_{v}(\sup \|\{y_{u^0_\delta,v}[\vartheta] - z_{u^0_\delta,v}[\vartheta]\}_m\|)\right) =$$

$$= \max_{v_\delta}\left(\lim_{\delta\to 0}\inf_{u}(\inf \|\{y_{u,v_\delta}[\vartheta] - z_{u,v_\delta}[\vartheta]\}_m\|)\right) =$$

$$= \lim_{\delta\to 0}\inf_{u}(\inf \|\{y_{u,v^0_\delta}[\vartheta] - z_{u,v^0_\delta}[\vartheta]\}_m\|).$$

Here u_δ and v_δ are controls whose realizations change their values only at the instants $t = \tau_k$ ($\tau_{k+1} - \tau_k = \delta > 0$), on the basis of the realizing positions $\{\tau_k, y[\tau_k], z[\tau_k]\}$. In this case the extremal controls u^0_δ and v^0_δ for $\tau_k \le t < \tau_{k+1}$ are constructed from the condition of aiming, at the instant $t = \tau_k$, at the point $q^*[\tau_k]$ and $p^*[\tau_k]$, respectively.

Uniqueness of the point $p^*[\tau]$ is not essential for the roughness of the case. In the linear case only the uniqueness of the vector $l^*[\tau]$ satisfying conditions (9) is important.

The convexity of the region $G^{(2)}_0$ also, generally speaking, plays no decisive role, and in the arguments this region may be replaced by its convex hull.

Finally, it should be noted that an effective verification of the roughness conditions stated in the article for those systems (1), (2) where the functions $f^{(i)}$ and $B^{(i)}$ are nonlinear is, in nontrivial cases, very difficult.

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Note: Figure translations are in progress. See original paper for figures.