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Abstract

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MATHEMATICS

A. Kh. GUDIEV

THE CAUCHY PROBLEM FOR NORMALLY HYPERBOLIC EQUATIONS WITH UNBOUNDED COEFFICIENTS

(Presented by Academician S. L. Sobolev on 10 V 1967)

In the monograph (1), S. L. Sobolev established conditions for the existence of a generalized solution and conditions for the existence of an m-times continuously differentiable solution of the Cauchy problem for equations of normally hyperbolic type. In the proof of these results the embedding theorem is used essentially, which does not allow one to weaken the stated conditions on the coefficients of the equations in terms of L_p-spaces.

In the present paper we consider a new scale of Banach spaces, different from the L_p-scale. For this scale, in terms of the spaces of this scale, conditions are given for the existence of a generalized solution and conditions for the existence of an m-times continuously differentiable solution of the Cauchy problem. These conditions in the scale under consideration are limiting.

Consider a two-dimensional plane, and let $\Omega_k^{(\varepsilon)}$ be the set of points (r_1, r_2) of the two-dimensional plane which we define by the equality

$$\Omega_k^{(\varepsilon)} = \left\{ (r_1, r_2);\ \begin{array}{c} kr_1r_2 - (n-s)r_1 - sr_2 - \varepsilon = 0, \\ \infty > r_1 > \begin{cases} 1, & \text{if } s < k, \\ s/k, & \text{if } s \geq k, \end{cases} \\ \infty > r_2 > \begin{cases} 1, & \text{if } n-s < k, \\ (n-s)/k, & \text{if } n-s \geq k, \end{cases} \end{array} \right\},$$

where $\varepsilon \geq 0$; k is a positive rational number; s is a fixed natural number not exceeding n.

Let D be a bounded domain in n-dimensional Euclidean space. Denote

$$D_1 = D \cap (\mathbf{x}_{n-s} = \text{const}); \qquad D_2 = \text{pr}_{R^{n-s}}\, D; \qquad \mathbf{x} = (x_1, \dots, x_n) = (\mathbf{x}_s, \mathbf{x}_{n-s});$$

R^{n-s} is the $(n-s)$-dimensional space of vectors $\mathbf{x}_{n-s}$.

Introduce into consideration the Banach space with mixed norm $L_{(r_1,r_2)}(D)$. We define the norm in $L_{(r_1,r_2)}(D)$ by the equality

$$\|f\|_{L_{(r_1,r_2)}D} = \left\|\|f\|_{L_{r_1}(D_1)}\right\|_{L_{r_2}(D_2)} < \infty.$$

Lemma 1. If (r_1, r_2) and $(\bar{r}_1, \bar{r}_2)$ are two distinct points belonging to $\Omega_k^{(\varepsilon)}$, then

$$(L_{(r_1,r_2)}(D) \cup L_{(\bar{r}_1,\bar{r}_2)}(D)) \setminus L_{(r_1,r_2)}(D) \neq \emptyset,$$

$$(L_{(r_1,r_2)}(D) \cup L_{(\bar{r}_1,\bar{r}_2)}(D)) \setminus L_{(\bar{r}_1,\bar{r}_2)}(D) \neq \emptyset.$$

Lemma 2. If $r_i \geq 1,\ i = 1, 2$, then

$$L_{\max(r_1,r_2)} \subset L_{(r_1,r_2)} \subset L_{\min(r_1,r_2)}.$$

Lemma 3. If $r_i \geq 1,\ i = 1, 2$, then $(L_{(r_1,r_2)} \cup L_p) \setminus L_p \neq \emptyset$ for any p satisfying the inequalities

$$\min(r_1, r_2) < p < \max(r_1, r_2).$$

Theorem 1. If D is an n-dimensional domain, star-shaped with respect to some ball, and $(r_1, r_2) \in \Omega_2^{(\varepsilon)}$, $p_i = 2r_i'$, $1/r_i + 1/r_i' = 1$, then the embedding

$$W_2^{(1)}(D) \to L_{(p_1,p_2)}(D')$$

takes place.

In the case $\varepsilon > 0$ the embedding operator is completely continuous.

Lemma 4. For domains star-shaped with respect to a certain ball, one has

$$W_2^1(D) \subset \bigcap_{\substack{p_i=2r_i \\ (r_1,r_2)\in\Omega_2^{(\varepsilon)}}} L_{(p_1,p_2)}(D').$$

Consider the equation

$$Lu \equiv \sum_{j=1}^{n}\sum_{i=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial^2 u}{\partial t^2} = F - \sum_{1}^{n} B_i \frac{\partial u}{\partial x_i} - Cu, \tag{1}$$

where A_{ij}, B_i, C, F are given functions of the point $(x, t) = (x_1, \ldots, x_n, t)$. Suppose, furthermore, that at every point of space and at every instant of time

$$A(p) \equiv \sum\sum A_{ij} p_i p_j > c \sum p_i^2, \qquad A_{ij} = A_{ji}, \tag{2}$$

where $c > 0$ is a constant.

We shall seek a solution of this equation satisfying the conditions

$$u\big|_{t=0} = u_0(x), \qquad \partial u/\partial t\big|_{t=0} = u_1(x). \tag{3}$$

I. **Statement of the generalized Cauchy problem** (see ([1])). Find a generalized solution of equation (1) which, for any value of t, is an element of the space W_2', while $\partial u/\partial t$, for any value of t, is an element of the space $L_2 = W_2$. The trajectory in W_2' and W_2, defined as the pair of functions u and $\partial u/\partial t$ in this pair of spaces, must be continuous and satisfy the initial conditions (3).

II. **Statement of the classical Cauchy problem.** Find a solution of equation (1) with conditions (3), having everywhere in the domain continuous derivatives of second order.

Consider a domain S in the plane $t = 0$. Let $S \times [0, T]$ be the cylinder constructed on S and bounded by the planes $t = 0$ and $t = T > 0$. Put $c_1 = n \max_{i,j(S\times[0,T])} |A_{ij}|$. Draw through the boundary of the domain S a straight cone S_3 so that for it $\cos^2 nt > c_1/(1 + c_1)$. The truncated cone lying in the cylinder $S \times [0, T]$ will be denoted by Ω^* (Ω^* is called a fundamental domain), and $\Omega(t)$ is the section of the fundamental domain by the plane $t = \text{const}$.

For clarity in what follows we give some notations and estimates from ([1]):

$$\int_{\Omega(t)} (L\omega)^2 \, d\Omega = L(t) < L_0, \qquad \int_{\Omega(t)} \omega^2 \, d\Omega = K_0(t \mid \omega),$$

$$\int_{\Omega(t)} \left[\sum \omega_{x_i}^2 + \omega_t^2\right] d\Omega = K_1(t \mid \omega),$$

$$\int_{\Omega(t)} \left[\sum\sum \omega_{x_i x_j}^2 + \alpha \sum \omega_{x_i t}^2 + \omega_{tt}^2\right] d\Omega = K_2(t \mid \omega).$$

Theorem 2. Suppose:

I. The coefficients A_{ij} are continuous with their first derivatives and satisfy the conditions

$$\max\{|A_{ij}|,\ |\partial A_{ij}/\partial x_k|,\ |\partial A_{ij}/\partial t|\} \le A(t) < A_0,$$

where A_0 is a certain fixed number.

II. The first generalized derivatives of B_i exist and, together with $|B_i|$, satisfy the conditions

$$\max\left\{|B_i|, \left\|\sum\left(\frac{\partial B_i}{\partial x_k}\right)^2+\left(\frac{\partial B_i}{\partial t}\right)^2\right\|_{L_{(r_1,r_2)}(\Omega(t))}\right\}\le A(t)<A_0,$$

where $(r_1, r_2)\in\Omega_2^{(\varepsilon)},\ \varepsilon>0$.

III. The first generalized derivatives of c exist and satisfy the conditions

$$\max\left\{|c|, \left\|\sum\left|\frac{\partial c}{\partial x_i}\right|+\left|\frac{\partial c}{\partial t}\right|\right\|_{L_p(\Omega(t))}\right\}\le A(t)<A_0,\quad p>2,$$

if $n=2,3$, and

$$\max\left\{\|c\|_{L_{(2r_1,\,2r_2)}}, \left\|\sum\left|\frac{\partial c}{\partial x_i}\right|+\left|\frac{\partial c}{\partial t}\right|\right\|_{L_{(r_1,\,r_2)}(\Omega(t))}\right\}<A_0$$

for $n\ge 4$, where $(r_1,r_2)\in\Omega_2^{(\varepsilon)}\cap\{r_1,r_2>2\}$.

IV. The generalized first derivatives of F exist and satisfy the conditions

$$\max\left\{\|F\|_{L_2(\Omega(t))}, \left\|\left[\sum\left(\frac{\partial F}{\partial x_i}\right)^2+\left(\frac{\partial F}{\partial t}\right)^2\right]^{1/2}\right\|_{L_2(\Omega(t))}\right\}\le F(t)<F_0.$$

V. $u_0\in W_2'$.

VI. $u_1\in L_2$.

Then there exists a solution of the Cauchy problem in the first formulation. The solution depends continuously on the initial data together with the first-order derivatives in the spaces W_2' and W_2.

The proof of Theorem 2 is carried out in the same way as in (1), except that instead of the embedding theorems of S. L. Sobolev we use Theorem 1. Let us trace what modifications must be made in the estimates (1) in order to cover the present Theorem 2.

Suppose the coefficients of equation (1) satisfy conditions I–IV. Replace in it all the functions A_{ij}, B_i, C, F by their mean values with respect to x, t, and the initial data by their mean values with respect to x. The new equation

$$Lu_h=F_h-B_{ih}(u_h)_{x_i}-C_hu_h$$

with the new conditions $u_h|_{t=0} = u_{0h}, \ (u_t)_h|_{t=0} = (u_1)_h$ has a solution u_h.

For the proof of the theorem it is sufficient to establish the validity of the estimates:

$$K_0(t \mid u_h) \leq K_0(0 \mid u_h) + 2\int_0^t K_0(t_1 \mid u_h)^{1/2} K_1(t_1 \mid u_h)^{1/2}\, dt_1, \tag{4}$$

$$K_1(t \mid u_h) \leq c\left\{K_1(0 \mid u_h) + \int_0^t K(t_1 \mid u_h)^{1/2}\left[F(t_1) + K_0(t_1 \mid u_h)^{1/2} + K_1(t_1 \mid u_h)^{1/2}\right] dt_1\right\}, \tag{5}$$

$$K_2(t \mid u_h) \leq M\left\{K_2(0 \mid u_h) + \int_0^t K_2(t_1 \mid u_h)^{1/2}\left[F(t_1) + K_0(t_1 \mid u_h)^{1/2} + \right.\right.$$

$$\left.\left. + K_1(t_1 \mid u_h)^{1/2} + K_2(t_1) u_h)^{1/2}\right] dt_1\right\}. \tag{6}$$

Estimate (4) does not depend on the coefficients of the equation and was established in (1). To establish (5), we estimate the expression

$$L(t_1)^{1/2} = \|Lu_h\|_{L_2(\Omega(t))} = \|F_h - B_{ih}(u_h)_{x_i} - C_h u_h\|_{L_2(\Omega(t))}.$$

If condition II is taken into account, then

$$\|B_i h u_{x_{ih}}\|_{L_2(\Omega(t))} \leq cK_1(t \mid u_h)^{1/2}.$$

For the estimate of $\|C_h u_h\|_{L_2}$ in the case $n \geq 4$, we use Theorem 1; then

$$\|C_h u_h\|_{L_2} \leq \|C_h\|_{L_{(2r_1,\,2r_2)}} \|u_h\|_{L_{(p_1,\,p_2)}} \leq c\|u_h\|_{W_2'} \leq c\left(K_0(t \mid u_h)^{1/2} + K_1(t \mid u_h)^{1/2}\right).$$

(In the case $n = 2, 3$, $\|C_h u_h\|_{L_2}$ is estimated in the same way as in (1).)

Taking these estimates into account, we obtain

$$L(t)^{1/2} \leq c\left(K(t \mid u_h)^{1/2} + K_1(t \mid u_h)^{1/2} + F(t)\right). \tag{7}$$

Using the estimate

$$K_1(t \mid u_h) \leq c\left\{K_1(0 \mid u_h) + \int_0^t \left[K_1(t_1 \mid u_h)^{1/2} + K_1(t_1 \mid u_h)^{1/2} L(t_1)^{1/2}\right] dt\right\}, \tag{8}$$

established by S. L. Sobolev, and the estimate (7) obtained above, we obtain (5).

To obtain estimate (6) we proceed as follows: we differentiate with respect to t the equation averaged with respect to x_i. Applying inequality (8) separately to each derivative and adding the estimates obtained, we have

$$K_2(t \mid u_h) \leq c\Bigg\{K_2(0 \mid u_h) + \int_0^t \Bigg[K_2(t_1 \mid u_h) + K_2(t_1 \mid u_h)^{1/2} \times \left(\left\|\sum L\frac{\partial u_h}{\partial x_i}\right\|_{L_2} + \left\|L\frac{\partial u}{\partial t}\right\|_{L_2}\right)\Bigg] dt\Bigg\}. \tag{9}$$

Using Theorem 1, we establish

$$\left[\left\|\sum L\frac{\partial u_h}{\partial x_i}\right\|_{L_2} + \left\|L\frac{\partial u}{\partial t}\right\|_{L_2}\right] \leq c\left\{F(t) + K_0(t \mid u_h)^{1/2} + K_1(t \mid u_h)^{1/2} + K_2(t \mid u_h)^{1/2}\right\}. \tag{10}$$

From (9) and (10) we obtain estimate (6).

Let us clarify the conditions for the existence of smooth solutions. Let:

1. The coefficients A_{ij} and their first-order derivatives $\partial A_{ij}/\partial x_k$, $\partial A_{ij}/\partial t$ be bounded. Put

$$\max\left\{|A_{ij}|,\ |\partial A_{ij}/\partial x_k|,\ |\partial A_{ij}/\partial t|\right\} \leq A_0.$$

2. The derivatives of A_{ij} of orders $l = 2, \dots, k_1$ satisfy the conditions

$$A_{ij} \in L^{(l)}_{(r_1,r_2)}(\Omega(t)), \quad (r_1, r_2) \in \Omega^{(\varepsilon)}_{(l-1)/2} \cap \{r_1, r_2 > 2\}, \quad \|A_{ij}\|_{L^{(l)}_{(r_1,r_2)}(\Omega(t))} \leq A_0.$$

3. The derivatives of B_i of orders $l = 1, 2, \dots, k_1 - 1$ satisfy the conditions

$$B_i \in L^{(l)}_{(p_1,p_2)}(\Omega(t)), \quad (p_1, p_2) \in \Omega^{(\varepsilon)}_{l/2} \cap \{p_1, p_2 > 2\}, \quad \|B_i\|_{L^{(l)}_{(p_1,p_2)}(\Omega(t))} \leq A_0.$$

4. The derivatives of C of orders $l = 0, 1, 2, \dots, k_1 - 1$ satisfy the conditions

$$C \in L^{(l)}_{(q_1,q_2)}(\Omega(t)), \quad (q_1, q_2) \in \Omega^{(\varepsilon)}_{l+1} \cap \{q_1, q_2 > 2\}, \quad \|C\|_{L^{(l)}_{(q_1,q_2)}(\Omega(t))} \leq A_0.$$

5. The derivatives of F of orders $l = 0, 1, 2, \dots, k_1 - 1$ satisfy the conditions

$$F \in L_2^{(l)}(\Omega(t)), \quad \|F\|_{L^{(l)}2(\Omega(t))} \le F(t) < F_0.$$

6. $u_0 \in W_2^{(k_1)}$.
7. $u_1 \in W_2^{(k_1-1)}$.

Theorem 3. In order that the Cauchy problem have a solution continuous together with its derivatives up to order m, it is sufficient that conditions 1-7 be satisfied for

$$k_1 \ge m + 1 + [n/2].$$

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

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REFERENCES

1. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Novosibirsk, 1962.

2. S. L. Sobolev, *Sur les équations aux dérivées partielles hyperboliques non-linéaires*, Roma, 1961.

Note: Figure translations are in progress. See original paper for figures.