THEOREMS ON HOMEOMORPHISMS IMPLEMENTED BY ELLIPTIC OPERATORS

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Abstract

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THEOREMS ON HOMEOMORPHISMS IMPLEMENTED BY ELLIPTIC OPERATORS

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Theorems on homeomorphisms for elliptic operators were established in various spaces in the works ($^{1-8}$) (see also (^{9}), pp. 170—177, 245—265, and the survey (^{10})); related questions were also studied in (^{11}). In the present paper a number of new theorems on homeomorphisms is established, and the relation between various theorems on homeomorphisms is studied. It turns out that many known theorems, and a number of new theorems, on homeomorphisms can essentially be obtained from the theorem of (^{5}) by means of a corresponding "gluing" of elements in the space of preimages and the space of images. We note that in (^{8}) the idea was expressed of the possibility of applying the gluing method to obtain new theorems on homeomorphisms.

1. Let G be a bounded domain of the space E_n, Γ its boundary, $\bar{G} = G \cup \Gamma$. In $\bar{G}$ let there be given a properly elliptic differential expression L of order $2m$ with complex coefficients, and on Γ a system of m differential expressions $\{B_j\}_{j=1}^{m}$ of orders m_j ($m_j \leqslant 2m - 1$), which we assume to be normal in the sense of Aronszajn—Milgram—Schechter (see (^{12,2,9})) and covering L. For simplicity we assume the coefficients of all differential expressions considered in the paper and the boundary Γ to be infinitely smooth. We introduce the necessary function spaces. Let $l \geq 0$ be an integer, $1 < p < \infty$, $p' = p/(p-1)$; $W_p^l(G)$ is the Sobolev space; $W_{p'}^{-l}(G)$ is the space conjugate to $W_p^l(G)$ with respect to $(u, v) = \int_G u\bar{v}\,dx$ (^{11,5});

$$\|u\|_{s,p} \text{ is the norm in } W_p^s(G) \quad (s \text{ an integer}).$$

If $l \geq 1$, then $W_p^{l-1/p}(\Gamma)$ denotes the completion of the set $C^\infty(\Gamma)$ in the norm $\langle\varphi\rangle_{l-1/p,p} = \inf \|u\|_{l,p}$, where the infimum is taken over all $u \in W_p^l(G)$ equal to φ on Γ; $W_{p'}^{-(l-1/p)}(\Gamma)$ is the space conjugate to $W_p^{l-1/p}(\Gamma)$ with respect to $\langle u, v\rangle = \int_\Gamma u\bar{v}\,dx$ (^{12,5}). If $s_0, s_1, \ldots, s_t$ are arbitrary integers, then we denote

$$K^{t+1}_{(s_0,\, s_j-1/p,\, p)} = W_p^{s_0}(G) + \sum_{j=1}^{t} W_p^{s_j-1/p}(\Gamma)$$

(here the upper index $t+1$ is equal to the number of summands in the direct sum). If
$a = (a_0, a_1, \dots, a_m) \in K^{m+1}_{(-s_0,\, -(s_j-1/p),\, p')}$, and
$v = (v_0, v_1, \dots, v_m) \in K^{m+1}_{(s_0,\, s_j-1/p,\, p)}$, then put

$$[a, v] = (a_0, v_0) + \sum_{j=1}^{m} \langle a_j, v_j \rangle.$$

Since the system of boundary expressions $\{B_j\}_{j=1}^m$ is, by assumption, normal, it can be supplemented by expressions $\{C_j\}_{j=1}^m$ to a Dirichlet system of order $2m$. But then there exists a "conjugate" system of boundary expressions $\{B'_j, C'_j\}_{j=1}^m$, also forming a Dirichlet system of order $2m$, such that Green' s formula holds (see ([12,2,9]))

$$(Lu, v) + \sum_{j=1}^{m} \langle B_j u, C'_j v \rangle = (u, L^+ v) + \sum_{j=1}^{m} \langle C_j u, B'_j v \rangle \qquad (u, v \in C^{2m}(\bar{G})). \quad (1)$$

Here the orders m_j, l_j, m'_j, l'_j of the expressions B_j, C_j, B'_j, C'_j satisfy the equality $m_j + l'_j = m'_j + l_j = 2m - 1$. For arbitrary $s \geq 0$ and $1 < p < \infty$ we define the operators $\mathcal{L}_{s,p}, \mathcal{L}^+_{s,p} : \mathcal{D}(\mathcal{L}_{s,p}) = \mathcal{D}(\mathcal{L}^+_{s,p}) = W_p^{2m+s}(G)$;

$$\mathcal{L}_{s,p} u = (Lu, B_1 u, \dots, B_{mu}) \in K_{(s,\, 2m+s-m_j-1/p,\, p)} \equiv K_{s,p}(G);$$

$$\mathcal{L}^+_{s,p} v = (L^+ v, B'_1 v, \dots, B'_{mv}) \in K^{m+1}_{(s,\, 2m+s-m'_j-1/p,\, p)} \equiv K'_{s,p}(G).$$

In this paper, for simplicity, we shall assume that the defect is absent*. Then the operators $\mathcal{L}_{s,p}$ and $\mathcal{L}^+_{s,p}$ map $W_p^{2m+s}(G)$ homeomorphically onto $K_{s,p}(G)$ and $K'_{s,p}(G)$, respectively.

Let now s be an arbitrary integer. By $\widehat{W}_p^s(G)$ ([4], [5], [8], [9]) we denote the completion of the set $C^\infty(\bar{G})$ in the norm

$$|||u|||_{s,p} = \left(\|u\|^p_{s,p} + \sum_{j=1}^{2m} \left\langle \frac{\partial^{j-1} u}{\partial \nu^{j-1}} \right\rangle^p_{s-j+1-1/p,p} \right)^{1/p} \qquad (\nu \text{ is the normal to } \Gamma). \quad (2)$$

It is clear that if $s \geq 2m$, then the norms $|||\ |||_{s,p}$ and $\|\ \|_{s,p}$ are equivalent; for $s < 2m$ they are not equivalent. The closure S of the mapping

$$u \to (u|_G, u|_\Gamma, \dots, \partial^{2m-1}u/\partial\nu^{2m-1}|_\Gamma) \quad (u \in C^\infty(\bar{G})),$$

considered as acting from $\widehat{W}_p^s(G)$ into $K^{2m+1}_{(s,\, s-j+1-1/p,\, p)}$ ($j = 1, \dots, 2m$; s arbitrary integer), maps isometrically $\widehat{W}_p^s(G)$ onto a subspace of the direct sum $K^{2m+1}_{(s,\, s-j+1-1/p,\, p)}$ (for $s \leq 0$, $S\widehat{W}_p^s(G) = K^{2m+1}_{(s,\, s-j+1-1/p,\, p)}$). The components of the vector Su will also be called the components of the element $u \in \widehat{W}_p^s(G)$. For every differential expression M of order $t \leq 2m$ defined in $\bar{G}$, the operator $u \to Mu$ ($u \in C^\infty(\bar{G})$) acts continuously from $\widehat{W}_p^s(G)$ to $W_p^{s-t}(G)$; similarly, for any boundary differential expression B of order $t \leq 2m-1$, the operator $u \to Bu|_\Gamma$ ($u \in C^\infty(\bar{G})$) acts continuously from $\widehat{W}_p^s(G)$ to $W_p^{s-t-1/p}(\Gamma)$ (s is any integer) (4, 5, 8, 9). If $u_0 \in \widehat{W}_p^s(G)$, then by Mu_0 (Bu_0) we denote the value, on the element u_0, of the closure of the mapping $u \to Mu$ ($u \to Bu|_\Gamma$) ($u \in C^\infty(\bar{G})$), considered as acting from $\widehat{W}_p^s(G)$ to $W_p^{s-t}(G)$ ($W_p^{s-t-1/p}(\Gamma)$). In accordance with this convention, for arbitrary $u \in \widehat{W}_p^s(G)$ we still denote

$$Su = (u|_G, u|_\Gamma, \dots, \partial^{2m-1}u/\partial\nu^{2m-1}|_\Gamma)\,.$$

We also note that the norm (2) is equivalent to the norm (8)

$$\{u\}_{s,p} = \left(\|u\|^p_{s,p} + \sum_{j=1}^{m} \langle B_j u\rangle^p_{s-m_j-1/p,p} + \sum_{j=1}^{m} \langle C_j u\rangle^p_{s-l_j-1/p,p}\right)^{1/p}. \tag{3}$$

2. In the present paper the following is used essentially.

Theorem 1 (5) (see also (4, 8, 9)). *For every integer s, the closure $\mathcal{L}_{s,p}$ of the mapping*

$$u \to (Lu, B_1u, \dots, B_{mu}) \quad (u \in C^\infty(\bar{G})),$$

considered as acting from $\widehat{W}_p^{2m+s}(G)$ to $K_{s,p}(G)$, establishes a homeomorphism between these spaces.

The use of Theorem 1 for obtaining other theorems on homeomorphisms is based on the following simple lemmas.

Lemma 1. *Let B_1 and B_2 be Banach spaces, and let T be a linear operator mapping B_1 homeomorphically onto B_2; let E_1 be a subspace of B_1, and $E_2 = TE_1$. Then the operator T naturally defines a linear operator T' mapping the quotient space B_1/E_1 homeomorphically onto B_2/E_2.*

If Q_2 is a Banach space, $Q_2 \subset B_2$ (topological embedding), then $Q_1 = T^{-1}Q_2$ will be a linear (generally speaking, nonclosed) subset of B_1. Introduce in Q_1 the graph norm

$$\|x\|^T_{Q_1} = \|x\|_{B_1} + \|Tx\|_{Q_2}$$

($x \in Q_1$). With respect to this norm Q_1 becomes a Banach space, which we denote by Q_1^T. The restriction of the operator T to Q_1 will also be denoted by T.

Lemma 2. *The operator T maps Q_1^T homeomorphically onto Q_2. If R_2 is a linear submanifold of Q_2, dense in Q_2, then $R_1 = T^{-1}R_2$ is dense in Q_1^T. In this case the closure of the mapping $x \to Tx$ ($x \in R_1$), regarded as acting from Q_1^T to Q_2, establishes a homeomorphism $Q_1^T \to Q_2$.*

We shall first apply Lemma 1, assuming that $B_1 = \widehat{W}_p^{2m+s}(G)$, $B_2 = K_{s,p}(G)$ (s an integer), and T is the operator $\mathfrak{L}_{s,p}$ occurring in Theorem 1. Choosing the subspace E_1 in various ways and putting $E_2 = TE_1$, we obtain various theorems on homeomorphisms.

3. Let $E_1 = E^1_{2m+s,p}$ be the subspace of $\widehat{W}_p^{2m+s}(G)$ consisting of elements $u \in \widehat{W}_p^{2m+s}(G)$ whose first component is equal to zero. It is clear that $\widehat{W}_p^{2m+s}(G)/E_1 = W_p^{2m+s}(G)$. From Green' s formula (1) it follows easily that $E_2 = E^2_{s,p} = \mathfrak{L}_{s,p}E_1$ consists of those and only those $F \in K_{s,p}(G)$ for which

$$[F, V] = 0 \quad \big(V \in \mathfrak{M}' = \{(v, C_1'v, \dots, C_m'v) :\ v \in C^\infty(\overline{G}),\ B_j'v|_\Gamma = 0 \quad (j = 1, \dots, m)\}\big). \tag{4}$$

Thus the following theorem is valid.

Theorem 2. *For each integer s, the closure $\mathfrak{L}_{s,p}$ of the mapping*

$$u \to (Lu, B_1u, \dots, B_{mu}) \quad (u \in C^\infty(\overline{G})),$$

regarded as acting from $\widehat{W}_p^{2m+s}(G)/E_1 = W_p^{2m+s}(G)$ to $K_{s,p}(G)/E_2$, establishes a homeomorphism between these spaces.

Let us note that $K_{s,p}(G)/E_2$ is the space adjoint with respect to $[\cdot,\cdot]$ to the closure in $K^{m+1}_{(-s,-s-l_j'-1/p',\,p')}$ of the set $\mathfrak{M}'$ (see (^{13}), Ch. 4, §5, item 4).

4. Let now $E_1 = \widetilde{E}^1_{2m+s,p}$ be the subspace of $\widehat{W}_p^{2m+s}(G)$ consisting of elements $u \in \widehat{W}_p^{2m+s}(G)$ for which $u|_G = 0$, $B_{ju}|_\Gamma = 0$ ($j = 1, \dots, m$). From the equivalence of the norms (2), (3) it follows that $\widehat{W}_p^{2m+s}(G)/E_1 = T_p^{2m+s}(G)$ coincides with the completion of the set $C^\infty(\overline{G})$ in the norm

$$\left(\|u\|^p_{2m+s,p} + \sum_{j=1}^m \langle B_{ju}\rangle^p_{2m+s-m_j-1/p,p}\right)^{1/p}. \tag{5}$$

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* All the results of the paper are also valid in the presence of a defect. In this case, it is only necessary in the theorems to replace the spaces of images and preimages by their corresponding subspaces.

From Green' s formula (1) it follows easily that $F \in K_{s,p}(G)$ belongs to $E_2 = \widetilde{E}^2_{s,p} = \mathfrak{L}_{s,p}E_1$ if and only if $F = (f, 0, \dots, 0)$,

$$(f, v) = 0 \quad \left(v \in C^\infty(\mathrm{pr})^+ = \{v : \; v \in C^\infty(\overline{G});\; B'_j v|_\Gamma = 0 \; (j = 1, \dots, m)\}\right),$$

therefore

$$K_{s,p}(G)/E_2 = W^s_p(G)/M \dotplus \sum_{j=1}^m W_p^{2m+s-m_j-1/p}(\Gamma) = W'_{s,p}(\mathrm{pr})^+ \dotplus$$

$$\dotplus \sum_{j=1}^m W_p^{2m+s-m_j-1/p}(\Gamma),$$

where M is the subspace of $W^s_p(G)$ consisting of elements $f \in W^s_p(G)$ such that $(f, v) = 0$ $(v \in C^\infty(\mathrm{pr})^+)$, while $W'_{s,p}(\mathrm{pr})^+$ is the space adjoint with respect to $(\cdot, \cdot)$ to the closure in $W^{-s}_{p'}(G)$ of the set $C^\infty(\mathrm{pr})^+$. Thus the following is valid.

Theorem 3. *For each integer s, the closure $T_{s,p}$ of the mapping*

$$u \to (Lu, B_1 u, \dots, B_{mu}) \quad (u \in C^\infty(\overline{G})),$$

regarded as acting from $T^{2m+s}_p(G)$ to

$$W'_{s,p}(\mathrm{pr})^+ \dotplus \sum_{j=1}^m W_p^{2m+s-m_j-1/p}(\Gamma),$$

establishes a homeomorphism between these spaces.

Let us note that if $2m+s-m_j > 0$ $(j = 1, \dots, m)$, then the norm (5) is equivalent to the norm $\|u\|_{2m+s,p}$ and $T^{2m+s}_p(G) = W^{2m+s}_p(G)$; therefore Theorem 3 is a generalization of the Lions-Magenes theorem (see (10), theo-

theorem 6.22), established for the special case of Dirichlet boundary conditions. From Theorem 3 there also follows directly the theorem on homeomorphisms of the work [3]. With the aid of Lemma 1 one can also obtain from Theorem 1 the theorem on homeomorphisms of the work [8].

5. We shall now use Lemma 2 and Theorem 3 in order to obtain, for the case of integral s, the Lions-Magenes theorem on homeomorphisms (see [2], Theorem 5.4). Let in Lemma 2

$$B_1 = T^{2m+s}_p(G), \qquad B_2 = W'_{s,p}(\Gamma)^+ + \sum_{j=1}^m W_p^{2m+s-m_j-1/p}(\Gamma)$$

($s < 0$ an integer), and let T be the operator $T_{s,p}$ occurring in Theorem 3. Let

$$Q_2 = L_p(G) + \sum_{j=1}^m W_p^{2m+s-m_j-1/p}(\Gamma) = K_{(0, 2m+s-m_j-1/p, p)},$$

$$Q_1 = T_{s,p}^{-1} Q_2.$$

Introduce in Q_1 the graph norm

$$\|u\|_{Q_1^T} = \|u\|_{2m+s,p} + \|Lu\|_{0,p} + \sum_{j=1}^{m} \langle B_j u \rangle_{2m+s-m_j-1/p}. \qquad (6)$$

The completion of $C^\infty(\overline{G})$ with respect to the norm (6) will be denoted by $W_{L,\{B_j\}}^{2m+s,p}(G)$. From Lemma 2 it follows directly that

Theorem 4. *For every integer $s < 0$, the closure of the mapping $T_{s,p}$ of $u \mapsto (Lu, B_1 u, \dots, B_{mu})$, $u \in C^\infty(\overline{G})$, considered as acting from $W_{L,\{B_j\}}^{2m+s,p}(G)$ into $K_{(0,2m+s-m_j-1/p,p)}$, establishes a homeomorphism between these spaces.*

Replacing, in the argument used for the proof of Theorem 3.1 of [2], the space $W_\gamma^{2m}(G)$ by $W_p^{2m+s}(G)$, it is easy to see that the norm (6) is equivalent to the norm

$$\|u\|_{W_L^{2m+s,p}(G)} = \|u\|_{2m+s,p} + \|Lu\|_{0,p}; \qquad (7)$$

therefore in Theorem 4 one may replace $W_{L,\{B_j\}}^{2m+s,p}(G)$ by $W_L^{2m+s,p}(G)$–the completion of the set $C^\infty(\overline{G})$ with respect to the norm (7), and the assertion of Theorem 4 coincides with the assertion (for integral s) of Theorem 5.4 of [2], proved under the additional assumption of uniqueness of the Dirichlet problem for the equation $Lu = f$.

With the aid of analogous arguments one can obtain from Theorem 3 and Lemmas 1, 2 (for integral s) the homeomorphism theorem 6.16 of [10].

6. Since Theorem 1 is also valid for nonintegral s [5], with the aid of Lemmas 1, 2 it is easy to obtain, for such s, assertions analogous to those proved above. Analogues of the theorems established are valid for operators generated by elliptic, in the sense of Petrovskii, systems of equations and normal boundary conditions [7]. They are also valid for operators generated by elliptic equations or systems with discontinuous coefficients and general boundary conditions and conjugation conditions [5, 7].

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Note: Figure translations are in progress. See original paper for figures.

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