APPROXIMATION OF DISCONTINUOUS FUNCTIONS BY FOURIER SUMS

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Abstract

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APPROXIMATION OF DISCONTINUOUS FUNCTIONS BY FOURIER SUMS

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The deviation of the partial sums of the Fourier series for various classes of discontinuous 2π-periodic functions has been investigated by many authors ($^{1-5}$). One of the most general results was obtained by A. V. Efimov (^{4}):

$$\sup_{f\in W^r_\beta H_\omega} |f(x) - S_n[f;x]| = \frac{1}{n^2}\Omega^{(n)}\frac{\ln n}{n^r} + O\left(n^{-r}\omega\left(\frac{1}{n}\right)\right). \quad (1)$$

Here $r > 0$, $\beta \in (-2, 2]$, or $r = \beta = 0$; $f \in W^r_\beta H_\omega$,

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{\pi k^r} \int_{-\pi}^{\pi} \varphi(t) \cos\left[k(t-x) + \frac{\beta\pi}{2}\right] dt, \quad (2)$$

where $\varphi \in H_\omega = W^0_0 H_\omega$, i.e., the modulus of continuity $\omega(\varphi, \delta)$ of the function φ does not exceed the true majorant of moduli of continuity $\omega(\delta)$ (the idea of introducing the classes $W^r_\beta H_\omega$ belongs to S. B. Stechkin); $S_n[f;x]$ is the n-th partial sum of the trigonometric Fourier series of the function f; $\Omega^{(n)} =$

$$= \sup_{\varphi\in H_\omega} \int_{-\pi}^{\pi} \varphi(t) \sin nt\, dt;$$

the estimate of the remainder term depends only on r and β (see also (^{5})).

In this note we study the rate of approximation by Fourier sums of nonperiodic continuous functions defined on $[-\pi, \pi]$. It turns out that the presence of a jump of the first kind at the point π in the periodic continuation does not affect the rate of convergence of Fourier sums inside the interval $(-\pi, \pi)$.

Let us agree on notation: $H_\omega(\langle a, b\rangle)$ is the class of functions defined at least on $\langle a, b\rangle$, whose modulus of continuity on this interval does not exceed $\omega(\delta)$; H^s_ω is the class of functions satisfying the conditions

$$f \in H_\omega([-\pi,\pi]), \quad f(\pi) - f(-\pi) = s; \qquad \operatorname{si} y = \int_y^\infty \frac{\sin t}{t}\,dt.$$

Theorem 1. *If* $|s| < \omega(2\pi)$, *then for* $x \in [-\pi,\pi]$ *we have*

$$\sup_{f \in H_\omega^s} \left| f(x) - S_n[f;x] - \frac{s}{\pi} \operatorname{sign} x \cdot \operatorname{si}(\pi - |x|)n \right| =$$

$$= \pi^{-2}\Omega^{(n)} \ln n + O\left(\omega\left(\frac{1}{n}\right)\right),$$

where the estimate of the remainder term depends only on s *and* ω.

For the proof we shall need the following lemmas ([5]), which are modifications of the corresponding lemmas of A. V. Efimov ([4]).

Lemma 1. *If* $\varphi \in H_\omega([-\sigma,\sigma])$, $\varphi(0) = 0$, *then*

$$\int_{-\sigma}^{\sigma} \varphi(t) \frac{\sin(n+\frac{1}{2})t}{\sin t/2}\,dt = \frac{n}{\pi} \sum_{k=1}^{m} \frac{1}{k} \int_{x_k}^{x_{k+1}} [\varphi(t) + \varphi(-t)] \sin nt\,dt + O\left(\omega\left(\frac{1}{n}\right)\right),$$

where

$$m = \left[\frac{n\sigma}{2\pi} - \frac{1}{4}\right], \qquad x_k = \frac{4k-3}{2n}\pi, \qquad \sigma \leqslant \pi.$$

Lemma 2.

$$\sup_{\varphi \in H_\omega([x_k,x_{k+1}])} \int_{x_k}^{x_{k+1}} \varphi(t) \sin nt\,dt = \frac{1}{n}\Omega^{(n)}.$$

Lemma 3. There exists a $\psi \in H_\omega$ such that

$$\psi(x + 2\pi/n) = \psi(x), \qquad \psi(-x) = -\psi(x), \qquad |\psi(x)| \le \frac{1}{2}\,\omega\left(\frac{x}{n}\right),$$

$$\int_a^{a+2\pi/n} \psi(t) \sin nt\,dt = \frac{1}{n}\Omega^{(n)}.$$

Proof of Theorem 1. We shall assume $s, x \ge 0$. Choose $\sigma \in (0, \pi/2)$ so that $\omega(2\pi - 2\sigma) > s$. In what follows n is sufficiently large; in particular, $\omega(\pi/n) < \omega(2\pi - 2\sigma) - s$. First let $\pi - \sigma \le x \le \pi$. Introduce the 2π-periodic function g, defined on $(-\pi,\pi]$ by the formula $g(x) = f(x) - (s/2)\operatorname{sign} x$. Then

$$f(x) - S_n[f;x] = g(x) - S_n[g;x] + (s/2)(\operatorname{sign} x - S_n[\operatorname{sign};x]).$$

It is not difficult to verify that

$$\operatorname{sign} x - S_n[\operatorname{sign}; x] = (2/\pi)\operatorname{sign} x \cdot \operatorname{si}(\pi - |x|)n + O(1/n),$$

and $sO(1/n) = O(\omega(1/n))$. Therefore

$$\Delta(f) = f(x) - S_n[f; x] - (s/\pi)\operatorname{sign} x \cdot \operatorname{si}(\pi - |x|)n = g(x) - S_n[g; x] + O(\omega(1/n)).$$

It is clear that

$$g \in H_\omega((0,\pi]), \qquad g \in H_\omega([\pi, 2\pi)), \qquad g \in H_{2\omega}((0, 2\pi)).$$

Further,

$$g(x) - S_n[g; x] = \int_{-x}^{2\pi - x} [g(x) - g(x+t)] \frac{\sin(n+1/2)t}{2\pi \sin t/2}\, dt =$$

$$= \int_{-x}^{-\sigma} + \int_{-\sigma}^{\sigma} + \int_{\sigma}^{2\pi - x} = I_1 + I_2 + I_3.$$

Put

$$h(t) = \frac{g(x) - g(x+t)}{2\pi \sin(t/2)}.$$

It is easy to see that on the interval $[\sigma, 2\pi - x]$

$$|h(t)| \le \frac{\omega(\pi + \sigma)}{2\pi \sin \sigma/2} = nO(\omega(1/n)),$$

and $\omega(h, \delta) = O(\omega(\delta))$. Hence, if $\xi = 2\pi/(2n+1)$, then

$$I_3 = \int_{\sigma}^{2\pi - x} h(t) \sin(n+1/2)t\, dt = -\int_{\sigma - \xi}^{2\pi - x - \xi} h(t+\xi) \sin(n+1/2)t\, dt =$$

$$= \frac{1}{2}\left\{ \int_{\sigma}^{2\pi - x - \xi} [h(t) - h(t+\xi)] \sin(n+1/2)t\, dt + \left(\int_{\sigma}^{\sigma+\xi} + \int_{2\pi - x - \xi}^{2\pi - x} \right) h(t) \sin(n+1/2)t\, dt \right\},$$

whence

$$I_3 = O(\omega(1/n)).$$

Similarly,

$$I_1 = O(\omega(1/n)).$$

By Lemma 1,

$$I_2 = -\frac{n}{2\pi^2} \sum_{k=1}^{m} \frac{1}{k} \int_{x_k}^{x_{k+1}} [g(x+t) + g(x-t)] \sin nt\, dt + O\left(\omega\left(\frac{1}{n}\right)\right).$$

From these formulas it follows that

$$\Delta(f) = -\frac{n}{2\pi^2}\sum_{k=1}^{m}\frac{1}{k}\int_{x_k}^{x_{k+1}}[g(x+t)+g(x-t)]\sin nt\,dt + O\left(\omega\left(\frac{1}{n}\right)\right). \quad (3)$$

Let

$$k_0 = \left[n/2 - n\alpha/2\pi + \frac{3}{4}\right].$$

Then $g(x+\cdot) \in H_\omega([x_k, x_{k+1}])$ $(k = 1, 2, \dots, k_0 - 1, k_0 + 1, \dots, m)$; $g(x+\cdot) \in H_{2\omega}([x_{k_0}, x_{k_0+1}])$; $g(x-\cdot) \in H_\omega([x_k, x_{k+1}])$ $(k = 1, 2, \dots, m)$. Applying Lemma 2, we obtain

$$|\Delta(f)| \le \frac{1}{\pi^2}\Omega^{(n)}\sum_{k=1}^{m}\frac{1}{k} + O\left(\omega\left(\frac{1}{n}\right)\right) + \frac{1}{2\pi^2}\frac{1}{k_0}\Omega^{(n)} =$$

$$= \frac{1}{\pi^2}\Omega^{(n)}\ln n + O\left(\omega\left(\frac{1}{n}\right)\right), \quad (4)$$

Define the function f_0 as follows:

$$f_0(u) = \begin{cases} \psi(u-x-2\pi), & u \in [-\pi, \beta), \\ 0, & u \in [\beta, x+\sigma-2\pi), \\ \min\{\omega(u-x-\sigma+2\pi),\ s+2\lambda\}, & u \in [x+\sigma-2\pi, x-\sigma), \\ s+2\lambda, & u \in [x-\sigma, \alpha), \\ \psi(x-u)+s+\lambda, & u \in [\alpha, x), \\ \psi(u-x)+s+\lambda, & u \in [x, \pi]. \end{cases}$$

Here ψ is the function constructed in Lemma 3; $\lambda = \max\psi$; α is the point nearest on the right to $x-\sigma$ at which the function $\psi(x-\cdot)$ assumes its greatest value; β is the point nearest on the left to $x+\sigma-2\pi$ at which the function $\psi(\cdot - x)$ assumes its least value. It can be shown that $f_0 \in H_\omega^s$.

Write relation (3) for f_0:

$$\Delta(f_0) = -\frac{n}{2\pi^2}\sum_{k=1}^{m-1}\frac{1}{k}\int_{x_k}^{x_{k+1}} 2\psi(-t)\sin nt\,dt -$$

$$-\frac{n}{2\pi^2 m}\int_{x_m}^{x_{m+1}}[f_0(x+t)+f_0(x-t)]\sin nt\,dt + O\left(\omega\left(\frac{1}{n}\right)\right) =$$

$$= \frac{n}{\pi^2} \sum_{k=1}^{m-1} \frac{1}{k} \int_{x_k}^{x_{k+1}} \psi(t) \sin nt \, dt + O\left(\omega\left(\frac{1}{n}\right)\right).$$

But

$$\int_{x_k}^{x_{k+1}} \psi(t) \sin nt \, dt = \frac{1}{n} \Omega^{(n)},$$

whence

$$\Delta(f_0) = \frac{1}{\pi^2} \Omega^{(n)} \sum_{k=1}^{m-1} \frac{1}{k} + O\left(\omega\left(\frac{1}{n}\right)\right) = \frac{1}{\pi^2} \Omega^{(n)} \ln n + O\left(\omega\left(\frac{1}{n}\right)\right). \quad (5)$$

Relations (4) and (5) imply the assertion of the theorem.

Now let $x < \pi - \sigma$. Then $\sin(\pi - |x|)n = O(1/n)$. Consequently,

$$\Delta(f) = f(x) - S_n[f; x] + O(\omega(1/n)).$$

Arguments analogous to those carried out above give here

$$\Delta(f) \leq \pi^{-2} \Omega^{(n)} \ln n + O(\omega(1/n)).$$

We shall construct the extremal function differently depending on whether the inequality $\omega(x - \sigma + \pi) \geq s$ holds or the opposite one holds. In the first case put

$$f_1(u) = \begin{cases} \min\{\omega(x - \sigma - u),\ s\} - \lambda, & u \in [-\pi, x - \sigma), \\ -\lambda, & u \in [x - \sigma, \alpha'), \\ \psi(x - u), & u \in [\alpha', x), \\ \psi(u - x), & u \in [x, \beta'), \\ -\lambda, & u \in [\beta', \pi], \end{cases}$$

where α' is the point nearest from the right to $x - \sigma$ at which $\psi(x - \cdot)$ assumes its least value; β' is the point nearest from the left to $x + \sigma$ at which $\psi(\cdot - x)$ assumes its least value. In the second case we set

$$f_2(u) = \begin{cases} \omega(u+\pi), & u \in [\pi, x-\sigma), \\ \omega(x-\sigma+\pi), & u \in [x-\sigma, \alpha), \\ \psi(x-u)+\omega(x-\sigma+\pi)-\lambda, & u \in [\alpha, x), \\ \psi(u-x)+\omega(x-\sigma+\pi)-\lambda, & u \in [x, \beta'), \\ \omega(x-\sigma+\pi)-2\lambda, & u \in [\beta', x+\sigma), \\ \min\{\omega(u-2\sigma+\pi)-2\lambda, s\}, & u \in [x+\sigma, \pi], \end{cases}$$

where α and β' were defined above. It is not difficult to verify that

$$f_i \in H^s_\omega, \qquad \Delta(f_i) = \pi^{-2}\Omega^{(n)} \ln n + O\Big(\omega\left(\frac{1}{n}\right)\Big) \qquad (i = 1, 2).$$

The theorem is proved.

Corollary. On the interval $[-\pi+\varepsilon, \pi-\varepsilon]$, where $0 < \varepsilon < \pi$, one has

$$\sup_{f \in H^s_\omega} |f(x) - S_n[f; x]| = \frac{1}{\pi^2}\Omega^{(n)} \ln n + O\Big(\omega\left(\frac{1}{n}\right)\Big),$$

in which the remainder term also depends on ε.

Comparison of the last assertion with relation (1) shows that the presence of a jump discontinuity of the first kind does not affect the rate of convergence of the Fourier sums at points separated from the point of discontinuity.

In conclusion we give, without proof, the following facts.

Theorem 2. Let $W^r_\beta H^s_\omega$ be the class of functions representable in the form (2), where $\varphi \in H^s_\omega$, $|s| \le \omega(2\pi)$, $r > 0$. Then, uniformly with respect to $x \in (-\pi, \pi)$, the relation

$$\sup_{f \in W^r_\beta H^s_\omega} \left| f(x) - S_n[f; x] - \frac{s}{\pi}(\pi - |x|)^r \operatorname{sign} x \int_{(\pi-|x|)n}^{\infty} \frac{\sin\left(t - \frac{\beta\pi}{2}\operatorname{sign} x\right)}{t^{r+1}}\, dt \right| = \pi^{-2}\Omega^{(n)} n^{-r} \ln n + O\Big(n^{-r}\omega\left(\frac{1}{n}\right)\Big).$$

Remark 1. For $x = \pi$ we have

$$\sup_{f \in W^r_\beta H^s_\omega} \left| f(\pi) - S_n[f; \pi] + \frac{s}{\pi}\frac{\sin \beta\pi/2}{\pi n^r} \right| = \frac{1}{\pi^2}\Omega^{(n)} \frac{\ln n}{n^r} + O\Big(n^{-r}\omega\left(\frac{1}{n}\right)\Big).$$

Remark 2. Uniformly with respect to $x \in [-\pi+\varepsilon, \pi-\varepsilon]$ one has

$$\sup_{f\in W^r_\beta H^s_\omega} |f(x) - S_n[f;x]| = \frac{1}{\pi^2}\Omega^{(n)}\frac{\ln n}{n^r} + O\left(n^{-r}\omega\left(\frac{1}{n}\right)\right).$$

Theorem 3. Let the function f be given and differentiable on $[-\pi,\pi]$, and let $f' \in H_\omega([-\pi,\pi])$, $f(\pi) - f(-\pi) = s$. Then

$$f(x) - S_n[f;x] = \frac{s}{\pi}\,\mathrm{sign}\,x \cdot \mathrm{si}(\pi - |x|)\,n + O\left(\frac{|s| + \omega(\pi) + \omega(n^{-1})\ln n}{n}\right),$$

where the constant entering the O-term is absolute.

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Note: Figure translations are in progress. See original paper for figures.